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Analysis of non-local models through energetic formulations

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Abstract

Non-local formulations have been intensively applied for the past 20 years to control the strain localisation resulting from strain-softening constitutive laws. We propose a framework that encompasses most of these non-local approaches. It relies on an energetic interpretation of both equilibrium equations and constitutive relations for generalised standard materials. It allows a systematic exploration of the properties of these models. We focus here on three aspects: the existence of solutions when applied to strain-hardening laws, the effective localisation control when applied to strain-softening laws and the thermodynamic bases.

A quite extended analysis is led under some assumptions, the more restrictive being that all the internal variables are dealt with the same regularising operator. It appears that only few approaches bring effective regularisation. Unfortunately, they do not rely on thermodynamic bases so that the Clausius–Duhem inequality is not automatically fulfilled. An alternative consists in giving up the assumptions of the analysis. In that case, it seems that no general result can be stated. Moreover, an illustration in the context of brittle damage shows that slightly different formulations can result in strongly differentiated behaviour, even though some specific choices lead nevertheless to a suitable model.
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1. Introduction

Difficulties encountered with numerical simulation of structures whose constitutive materials exhibit softening (plasticity with softening, damage, coupled phenomena) led in the 1980s to interest for roughly speaking "non-local" models, hoping they would be able to restore stability properties in these simulations. The analysis was based on the recognition that models without an internal length would exhibit spurious

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mesh dependence, localisation effects towards bands of vanishing thickness and diffuse bifurcations. Many authors develop or reappraise non-local [1] approaches which can be classified in three families:

1. Incorporating in the classical continuum formulation either non-local or additional primal variables (non-local strain, higher strain gradients or directors) (Triantafyllidis and Aifantis, 1986; Frémond and Nedjar, 1996; de Borst and Sluys, 1991; Forest et al., 2000). These models require an extended version of the principle of virtual work.
2. Incorporating non-local or gradients of internal variables or modifying the yield function with non-local or gradients of hardening parameters (Bazant and Pijaudier-Cabot, 1988; Peerlings et al., 1996; Comi, 2001; Aifantis, 1987; Mühlhaus and Aifantis, 1991; de Borst and Mühlhaus, 1992; Svedberg and Runesson, 1997).
3. Incorporating non-local driving forces in the yield function (Pijaudier-Cabot and Bazant, 1987; Pijaudier-Cabot and Huerta, 1991; Comi and Perego, 2001).

In this paper an analysis of various types of non-local models will be conducted, starting from an energetic framework based on a generalisation of the variational property associated with generalised standard material formatted constitutive equations. Previous work has been performed for models with internal variable gradients (Lorentz and Andrieux, 1999). Here, we focus on the regularisation by several ways of some variables within the usual local constitutive relations, see also the review paper of Peerlings et al. (2001) for a different approach of similar questions. Again, the framework proposed will be shown to encompass several existing models or slight modifications of them. Some advantages can be gained from an energetic approach concerning the general setting of the model as well as its numerical implementation, as recalled now.

Even though finite element algorithms based on approximations of the displacement and the internal variable fields follow naturally from energetic formulations, they can also be derived for other types of approaches. Nevertheless, in the latter case, a global form of the consistency equation has to be derived (Simo, 1988), and leads to a variational equation for the plastic multiplier (see for instance Mühlhaus and Aifantis, 1991). For internal variable gradient models, a need for boundary conditions for the internal variable field arises and the choice is somewhat arbitrary: zero normal gradients are used in most papers without justifications, although Svedberg and Runesson (2000) remarked that this homogeneous condition is the only one which ensures that a bifurcation may arise from a homogeneous state. Besides, when not associated with an energy, non-local models generally lead to non-symmetric tangent operators which render their use more difficult in numerical computations and in the analysis of the conditions for localization, as stated by Pijaudier-Cabot and Huerta (1991). Finally, non-energy-based models cannot automatically ensure (at least) global fulfilment of the second thermodynamics law and examination of the stability of solutions or even selection of stable solutions cannot be performed.

Apart from these considerations on the formulation of the models itself, the energy approach will be used in this paper to lead a general analysis on the regularisation procedure by examining the conditions under which the minimisation problems are well-posed. The paper is organised as follows. First, a recall of energetic formulations for generalised standard constitutive relations is set out and extended to define a

[1] In order to fix the vocabulary, we will henceforth call "non-local" any operator or model which is weakly non-local in the sense defined by Rogula (1982), that is operators or models entering relations that are not invariant with respect to any scale transformation $\mathbf{x}' = a\mathbf{x}$ of the physical space. This notion of weak non-locality is related to the existence of one (or several) internal lengths while Rogula's definition of strict non-locality relies on the examination of the support of functions that are in relation via the operator or the model (a strictly local operator transforms any field in another field whose support domain is contained in the support domain of the original field). Therefore, gradient models usually belong to the class of weakly non-local ones while regularised models usually belong to the class of strictly non-local ones.

global constitutive equation suited for non-local models. Then, a general approach to regularise state variables is derived and a classification of different choices allowed by this approach is performed together with the corresponding variational properties. The subsequent chapters are devoted to the analyses of these different types of non-local models from the standpoint of the thermodynamic ground, the well-posedness of the initial boundary value problem and the regularisation efficiency through localisation features. Links with existing formulations or regularisation choices appeared in the literature are also detailed.

2. Energetic formulations

From now on, we restrict our attention to rate-independent generalised standard materials under isothermal conditions (see Halphen and Nguyen, 1975). Despite the restrictions, such a framework is appropriate to analyse regularised constitutive laws, thanks to its underlying mathematical basis. In addition, infinitesimal strains are assumed for the sake of simplicity, even though this hypothesis may be relaxed (see Lorentz and Cano, 2002) for an extension to finite strain.

Under these assumptions, the thermodynamic state of a material point is defined by the strain $\boldsymbol{\varepsilon}$ and a set of internal variables a. Derivation of the free Helmholtz' energy $\Phi(\boldsymbol{\varepsilon}, a)$ provides expressions for the stress $\boldsymbol{\sigma}$ and the set of driving forces A associated to a:

$$\boldsymbol{\sigma} = \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, a), \qquad A = -\frac{\partial \Phi}{\partial \alpha}(\boldsymbol{\varepsilon}, a) \tag{1}$$

The driving forces govern the rate of the internal variables $\dot{\boldsymbol{a}}$ through a flow rule and a consistency condition which rely on a convex threshold function $f(A)$:

$$\dot{\boldsymbol{a}} = \lambda \frac{\partial f}{\partial A}(A) \tag{2}$$

$$f(A) \leqslant 0, \quad \lambda \geqslant 0, \quad \lambda f(A) = 0 \tag{3}$$

λ is the plastic multiplier, in reference to plastic behaviour, and $\{A; f(A) \leqslant 0\}$ is the reversibility domain in the space of driving forces.

This section aims at showing that such a constitutive behaviour is equivalent to the minimisation of an incremental potential energy. Mathematical optimisation then provides a framework to study the properties of the mechanical response, namely the well or ill-posed character of the problem. Definitions and effects of regularisation can also be studied within this framework.

2.1. Incremental potential energy and related properties

To derive a minimisation principle from the constitutive equations above-mentioned, we proceed in three steps:

- introduction of the dissipation potential,
- extension of the state variables and the potentials to the structure scale,
- time discretisation.

First, the dissipation potential $\Delta(\dot{\boldsymbol{a}})$ is defined as the maximal dissipation for a given internal variable rate $\dot{\boldsymbol{a}}$:

$$\Delta(\dot{\boldsymbol{a}}) \underset{def.}{=} \max_{\substack{A \\ f(A)\leqslant 0}} A \cdot \dot{\boldsymbol{a}} \tag{4}$$

Eqs. (2) and (3) appear respectively as Euler's equation and Kuhn and Tucker's condition for the maximisation problem (4) (see for instance Ekeland and Temam, 1974). Moreover, Δ can be recognised as the Legendre–Fenchel transform or conjugate function of the indicator function of the reversibility domain defined by $f(A)\leqslant 0$ (hence, Δ is positive homogeneous of degree one, a characteristic of rate-independent materials). Therefore, thanks to the convexity of the function f, the evolution equations (2) and (3) are equivalent to:

$$\begin{cases} \dot{\boldsymbol{a}} = \lambda \dfrac{\partial f}{\partial A}(A), \\ f(A)\leqslant 0,\ \lambda \geqslant 0,\ \lambda f(A) = 0 \end{cases} \iff A \in \partial\Delta(\dot{\boldsymbol{a}}) \tag{5}$$

where an interpretation of the subgradient $\partial\Delta$ is given by:

$$A \in \partial\Delta(\dot{\boldsymbol{a}}) \iff \quad \forall \hat{\boldsymbol{a}}\ \ \Delta(\hat{\boldsymbol{a}}) \geqslant \Delta(\dot{\boldsymbol{a}}) + A \cdot (\hat{\boldsymbol{a}} - \dot{\boldsymbol{a}}) \tag{6}$$

Note that this equivalence clearly shows that generalised standard materials enforce a positive dissipation, i.e. fulfilment of the Clausius–Duhem inequality. Indeed, by setting $\hat{\boldsymbol{a}} = 0$ in (6), one obtains:

$$D \underset{def.}{=} \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}} - \dot{\Phi} = A\dot{\boldsymbol{a}} \geqslant \Delta(\dot{\boldsymbol{a}}) - \Delta(0) = \Delta(\dot{\boldsymbol{a}}) \geqslant 0 \tag{7}$$

Note also that thanks to a characteristic property of subgradients for convex conjugate functions:

$$A \in \partial\Delta(\dot{\boldsymbol{a}}) \iff \dot{\boldsymbol{a}} \in \partial\Delta^*(A) \tag{8}$$

where the conjugate function of the dissipation potential is the indicator function of the domain of reversibility:

$$\Delta^*(A) = \begin{cases} 0 & \text{if } f(A)\leqslant 0 \\ +\infty & \text{otherwise} \end{cases} \tag{9}$$

we easily derive the maximum dissipation principle of Hill, that is:

$$\begin{aligned} A \in \partial\Delta(\dot{\boldsymbol{a}}) \iff \dot{\boldsymbol{a}} \in \partial\Delta^*(A) &\iff \forall \widehat{A}\ \ \Delta^*(\widehat{A}) \geqslant \Delta^*(A) + \dot{\boldsymbol{a}} \cdot (\widehat{A} - A) \\ &\Downarrow \\ \forall \widehat{A}\ \text{ s.a. } f(\widehat{A})\leqslant 0 \quad & A \cdot \dot{\boldsymbol{a}} \geqslant \widehat{A} \cdot \dot{\boldsymbol{a}} \end{aligned} \tag{10}$$

The second step consists in leaving the material point scale for the structure scale. Consider now the fields of state variables, that is the strain field ε and the internal variable field a (note that the field is denoted in the same way as its pointwise value to avoid cumbersome notations). Following a proposal of Germain et al. (1983), the (global) free energy F and the dissipation potential D relative to the whole structure (body domain Ω) are defined by:

$$F(\boldsymbol{\varepsilon}, a) = \int_\Omega \Phi(\boldsymbol{\varepsilon}(x), a(x))\,\mathrm{d}x \tag{11}$$

$$D(\dot{\boldsymbol{a}}) = \int_\Omega \Delta(\dot{\boldsymbol{a}}(x))\,\mathrm{d}x \tag{12}$$

Keeping in mind the formalism for generalised standard materials, a *global* constitutive relation is derived from the potentials (11) and (12), in which the state variables are fields and the driving forces, including the stress, are linear forms operating on these fields:

$$\boldsymbol{\sigma} = \frac{\partial F}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, a), \qquad A = -\frac{\partial F}{\partial a}(\boldsymbol{\varepsilon}, a) \tag{13}$$

$$A \in \partial D(\dot{a}), \quad \text{i.e. } \forall \hat{a} \;\; D(\hat{a}) \geqslant D(\dot{a}) + \langle A | \hat{a} - \dot{a} \rangle \tag{14}$$

where $\langle \cdot | \cdot \rangle$ denotes the duality product. It can be shown that the global formulation (13) and (14) and the local one (1)–(3) are equivalent under some assumptions of regularity (Ekeland and Temam, 1974).

The global formulation seems to be the appropriate level to introduce some non-locality in a natural way: non-local or higher order gradient terms can easily be introduced in the potentials while some variables can be replaced by non-local counterparts, as will be shown later. Moreover, the formalism of generalised standard materials allows to retain thermodynamic properties, at least at the structural scale: as in the local constitutive equation (6) for standard materials, the choice of the global evolution law (14) is equivalent to a global maximum dissipation principle of Hill, a property used as a start point by Polizzotto and Borino (1998) and Benvenuti et al. (2002) to derive a set of local evolution equations. Maximal structural dissipation results in a positive dissipation of the structure, whatever the evolution of the state variables, thus fulfilling automatically the second principle of thermodynamics. Nevertheless, local properties may be lost; this is not surprising since non-locality introduces coupling between material points so that the principle of local action has to be given up. To retrieve a local positive dissipation, some non-locality residual (Edelen and Laws, 1971; Polizzotto and Borino, 1998) or an extra entropy flux (Maugin, 1990) have to be introduced. However, a local definition of the dissipation does not seem necessary as long as no thermal coupling is taken into account.

The third and last step requires a time discretisation because only the incremental (discretised) problem can generally be expressed as a minimum principle. Consider a given time step $[t^-, t]$; q^- and q denote respectively the value of a quantity q at the beginning and the end of the time step. Discretisation of Eqs. (13) and (14) by means of a backward Euler scheme leads to:

$$-\frac{\partial F}{\partial a}(\boldsymbol{\varepsilon}, a) \in \partial D(a - a^-) \tag{15}$$

If D is continuous at least in one point, which is usually the case, then (15) is equivalent to the following minimum problem (see once more Ekeland and Temam, 1974):

$$a = \arg\min_{a} E(\boldsymbol{\varepsilon}, a) \quad \text{with } E(\boldsymbol{\varepsilon}, a) = F(\boldsymbol{\varepsilon}, a) + D(a - a^-) \tag{16}$$

Integration of the global discretised constitutive law is expressed as a minimum of the incremental potential energy E with respect to the internal variable field. Note that E is named "incremental" because it depends explicitly on the internal variables at the beginning of the time step a^-.

Besides, the principle of virtual work reads:

$$\forall \delta\boldsymbol{\varepsilon} \in KA^0 \quad 0 = \int_\Omega \boldsymbol{\sigma}(x) \cdot \delta\boldsymbol{\varepsilon}(x) = \left\langle \frac{\partial F}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, a) \middle| \delta\boldsymbol{\varepsilon} \right\rangle \Longleftrightarrow \boldsymbol{\varepsilon} = \arg\min_{\boldsymbol{\varepsilon} \in KA} E(\boldsymbol{\varepsilon}, a) \tag{17}$$

where KA and KA^0 denote respectively the affine manifold of admissible strain fields (i.e. a strain field that is compatible with a displacement field which respects the Dirichlet boundary conditions) and its associated vector space. In this expression, we have implicitly assumed that there is no external work (no applied force). This is a choice for the sake of simplicity. Else, the results to come would be preserved but would also require to be expressed in terms of displacements instead of strains and the potential energy should include the external work (as in classical elastostatics). Finally, the equilibrium equations are expressed as a minimum of the incremental potential energy E with respect to the strain field.

Were the incremental potential energy convex, coercive and lower semi-continuous, then both minimisations (16) and (17) would result in a single one with respect to the couple $(\boldsymbol{\varepsilon}, a)$ and would admit a unique solution:

$$(\boldsymbol{\varepsilon}^*, a^*) = \underset{\substack{\boldsymbol{\varepsilon} \in KA \\ a}}{\arg\min} E(\boldsymbol{\varepsilon}, a) \tag{18}$$

These properties are intimately related to strain hardening (see Marigo, 2002). In the case of strain softening, the potential energy is no more convex and (16) and (17) cannot be expressed a priori as a minimum with respect to the pair $(\boldsymbol{\varepsilon}, a)$. Nevertheless, Francfort and Marigo (1993) proposed to preserve the minimum principle (18) as a characterisation of *stable* solutions. Unfortunately, they also showed in the context of a simple damage model that there is no stable solutions because the potential energy is no more lower semi-continuous.

Obviously, to retrieve a well-posed problem, i.e. existence of a finite number of solutions, the potential energy should be altered, either its expression or the meaning of its variables.

2.2. Refinement of the potential energy

Since minimisation (18) should characterise stable solutions of the strain-softening problem and nevertheless fails to retain any solution, an essential ingredient is surely missing in the local constitutive behaviour and therefore in the global one too.

Led by a mathematical point of view, Francfort and Marigo (1993) departed from the lack of required properties of the potential energy. The fact that convexity is not fulfilled is not surprising since bifurcations are expected in strain-softening problems (see Benallal et al., 1993 for a review). But the questions of coerciveness and lower semi-continuity are more troublesome. To avoid the first one, these authors focused on a partial damage law: the material cannot be totally damaged, some stiffness remains even at the ultimate stage. Therefore, coerciveness is preserved, thanks to the residual stiffness. The second one is dealt with a mathematical tool: the potential energy is replaced by its lower semi-continuous envelop (quasi-convexification of its integrand) so that the minimisation problem (18) is relaxed. In that way, solutions are retrieved. They can be interpreted as a fine mixture of sound and damaged layers that organise themselves to minimise the energy. However, even though attractive, such a regularisation strategy suffers from the severe restriction of partial damage and can therefore be applied in few cases only. Besides, tractable expressions of the quasi-convexified are not available for 3D problems.

An alternative approach is based on physical considerations. In the context of strain-softening materials, strain localisation is observed experimentally and therefore does not appear as a spurious artefact of the model (see, for instance, Desrues and Chambon, 1985 for experiments on soils). When it occurs, the length scale of the macroscopic mechanical fields becomes of the same magnitude as the microstructural length scale, thus precluding the main homogenisation hypothesis of scale separation: classical homogenisation is no more valid to derive the local constitutive laws. Several authors proposed to enrich the homogenisation schemes to include this length scale interaction. When the interaction is given a priori, as for materials with graded properties, Drugan and Willis (1996) succeeded in deriving a homogenised non-local constitutive law. However, the task becomes still harder when the interaction results from the evolution of the macroscopic fields themselves, as for localisation: then, only rough approximations have been obtained so far. Thus, in the context of hyperelasticity, Bardenhagen and Triantafyllidis (1994) introduced strain gradient terms into the energy based on the analysis of a periodic lattice. For ductile damage, Gologanu et al. (1997) enhanced Gurson's model by taking into account gradient effects on the boundary of the elementary representative volume; it also resulted in the introduction of strain gradient terms. Andrieux et al. (1996) proposed a homogenisation scheme to get some guidelines for the introduction of the gradient of internal

variables into the energy, based on the replacement of Taylor's scheme by a first order variation of the macroscopic internal variable field within a patch of elementary representative volumes. Other authors preferred to introduce scale effects on the basis of phenomenological propositions, mainly in the context of metal plasticity where the mechanism of dislocation interaction is well studied (see, for instance, Gao et al., 1999; Fleck and Hutchinson, 1993; Forest et al., 2000). Whatever their basis, all these approaches result in the introduction of some characteristic lengths of the microstructure in the macroscopic equations.

A link between both classes of approaches (mathematical relaxation vs. explicit interaction of the microstructures) has been established by Lorentz (1999) for some models with internal variable gradients: when the characteristic length goes to zero, the gradient model converges (in the sense of Γ-convergence) towards the relaxed model, provided that a residual stiffness remains.

2.3. Regularisation of the state variables

An alternative to the refinement of the potential energy consists in substituting a smoothed variable or a smoothed driving force for its local counterpart into the state equations, the flow rule or the consistency condition. This pragmatic way of introducing some regularisation has been used by several authors in the past and is still intensively applied, especially in numerical simulations of damage. The purpose is evidently to smooth up the abrupt spatial variations of the mechanical fields observed in localisation bands. In the framework of the energetic approach, that consists in keeping the expression of the energy unchanged but altering the meaning of the state variables within the potential energy. Such a regularisation can be achieved in two steps.

First, a regularising operator $\mathbf{R}$, acting on the strain field or the internal variable field, has to be chosen. Consider for instance the internal variable field a. This operator produces the smoothed internal variable field, say $\bar{a}$, which is an element of a new space of functions $\mathbf{V}_\mathbf{R}$, desirably "smoother" than the functions of the space of the original internal variable fields $\mathbf{V}$:

$$\begin{aligned} &\mathbf{R} : \mathbf{V} \to \mathbf{V}_\mathbf{R} \subset \mathbf{V} \\ &\quad a \mapsto \bar{a} = \mathbf{R}a \end{aligned} \tag{19}$$

The definition of the operator $\mathbf{R}$ may rely on an integral as well as a differential characterisation, see for instance the frequently used Gaussian convolution operator $\mathbf{M}$, introduced by Eringen (1972) in the context of non-local elasticity, or the gradient penalty operator $\mathbf{P}$ (Peerlings et al., 1996), which both involve an internal length L_c:

$$(\mathbf{M}a)(x) = \bar{a}(x) = \int_\Omega \omega(x,y)a(y)\,\mathrm{d}y \quad \text{with} \begin{cases} \omega(x,y) = \dfrac{1}{V_r(x)} \exp(-(\|x-y\|^2)/2L_c^2) \\ V_r(x) = \int_\Omega \exp(-(\|x-y\|^2)/2L_c^2)\,\mathrm{d}y \end{cases} \tag{20}$$

$$\mathbf{P}a = \bar{a} \text{ solution of } \begin{cases} \bar{a} - L_c^2 \Delta \bar{a} = a & \text{in } \Omega \\ \nabla \bar{a} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases} \tag{21}$$

Two natural demands arise on regularising operators: they must leave unchanged the constant fields and be injective. The latter demand is based on the physical consideration that the "true" state variable which is used for the description of the local thermodynamic state of the material is the non-smoothed one, so that it must be determined uniquely from the smoothed one. Remark also that the linearity of operator $\mathbf{R}$ is a matter of convenience which ensures in particular that the regularisation involved in the rate constitutive equations is the same (differentiation of a linear operator leads to itself).

The second step consists in substituting the smoothed state variable for its local counterpart into part or all the potential energy. Depending whether the strain or the internal variables are smoothed, the process is

Table 1
Different types of energy regularisations

	Strain regularisation	Internal variable regularisation
Coupled regularisation	$(\boldsymbol{\varepsilon}^*, a^*) = \arg\min_{\boldsymbol{\varepsilon},a} E(\mathbf{R}\boldsymbol{\varepsilon}, a)$ $\Downarrow$ $\begin{cases} \boldsymbol{\varepsilon}^* = \arg\min_{\boldsymbol{\varepsilon}} E(\mathbf{R}\boldsymbol{\varepsilon}, a^*) \\ a^* = \arg\min_{a} E(\mathbf{R}\boldsymbol{\varepsilon}^*, a) \end{cases}$	$(\boldsymbol{\varepsilon}^*, a^*) = \arg\min_{\boldsymbol{\varepsilon},a} E(\boldsymbol{\varepsilon}, \mathbf{R}a)$ $\Downarrow$ $\begin{cases} \boldsymbol{\varepsilon}^* = \arg\min_{\boldsymbol{\varepsilon}} E(\boldsymbol{\varepsilon}, \mathbf{R}a^*) \\ a^* = \arg\min_{a} E(\boldsymbol{\varepsilon}^*, \mathbf{R}a) \end{cases}$
Split regularisation		$(\boldsymbol{\varepsilon}^*, a^*) = \arg\min_{\boldsymbol{\varepsilon},a} F(\boldsymbol{\varepsilon}, \mathbf{R}a) + D(a - a^-)$ $\Downarrow$ $\begin{cases} \boldsymbol{\varepsilon}^* = \arg\min_{\boldsymbol{\varepsilon}} F(\boldsymbol{\varepsilon}, \mathbf{R}a^*) + D(a^* - a^-) \\ a^* = \arg\min_{a} F(\boldsymbol{\varepsilon}^*, \mathbf{R}a) + D(a - a^-) \end{cases}$
Uncoupled regularisation	$\begin{cases} \boldsymbol{\varepsilon}^* = \arg\min_{\boldsymbol{\varepsilon}} E(\boldsymbol{\varepsilon}, a^*) \\ a^* = \arg\min_{a} E(\mathbf{R}\boldsymbol{\varepsilon}^*, a) \end{cases}$	$\begin{cases} \boldsymbol{\varepsilon}^* = \arg\min_{\boldsymbol{\varepsilon}} E(\boldsymbol{\varepsilon}, \mathbf{R}a^*) \\ a^* = \arg\min_{a} E(\boldsymbol{\varepsilon}^*, a) \end{cases}$

respectively named strain regularisation or internal variable regularisation (it could also be a combination of both):

- Strain regularisation: $\boldsymbol{\varepsilon} \to \mathbf{R}\boldsymbol{\varepsilon}$
- Internal variable regularisation: $a \to \mathbf{R}a$

Several classes of models are obtained depending whether both minimisation problems with respect to the strain and the internal variables respectively are concerned (denoted as coupled regularisation) or not (denoted as uncoupled regularisation). Besides, the smoothed variable may be introduced into part of the potential energy only, for instance the free energy and not the dissipation potential: this approach will be named split regularisation. These various formulations are gathered in Table 1.

Each of them will lead to specific regularisation properties studied in the next sections, under the following assumptions:

- rate-independent generalised standard materials,
- isothermal conditions,
- infinitesimal strains,
- no external work (only to simplify the presentation),
- linear injective regularising operator that leaves constant fields unchanged, [2]
- when smoothing the set of internal variables, the same operator is applied on all of them.

[2] The examples which highlight the analysis will be based without much loss of generality on the gradient penalty regularising operator $\mathbf{P}$ as defined in (21) because of its attractive properties. We just mention here that this operator has its range included in $H^1(\Omega)$, is self-adjoint and fulfils the demands of being injective, linear and leaving constant fields unchanged. For more details, a mathematical study is presented in Appendix A.

Three aspects are examined.

Effect on a well-posed problem: Applying a regularisation technique to a local problem which is initially well-posed should result in a still well-posed non-local problem.

Thermodynamics: As we focus on generalised standard materials, the Clausius–Duhem inequality is automatically fulfilled by the local model. The regularised one should enjoy the same property.

Effective regularisation: As soon as the local model exhibits strain softening, it may become ill-posed, in relation with the fact that no length scale is selected for the localisation bands. Therefore, the regularisation method could be considered effective as soon as the non-local model does select a minimal (non-zero) width. The study of this selected band width will be achieved by a localisation analysis which is fully described in Appendix B for the sake of completeness.

3. Coupled regularisation

A first class of regularised models, corresponding to the first line in Table 1, is studied in this section. A state variable is replaced by its smoothed counterpart into the potential energy, while the solution of the problem is still given by a minimisation with respect to both the strain field and the internal variable field:

$$(\boldsymbol{\varepsilon}^*, a^*) = \arg\min_{\boldsymbol{\varepsilon},a} E_R(\boldsymbol{\varepsilon}, a) \quad \text{with} \begin{cases} E_R(\boldsymbol{\varepsilon}, a) = E(\mathbf{R}\boldsymbol{\varepsilon}, a) \\ \text{or} \\ E_R(\boldsymbol{\varepsilon}, a) = E(\boldsymbol{\varepsilon}, \mathbf{R}a) \end{cases} \tag{22}$$

The denomination "coupled regularisation" is clear: the minimisation is performed with respect to a smoothed variable, thus coupled with the regularisation. Of course, the properties of the function to be minimised are altered by the regularising operator, as expected when dealing with ill-posed problems. However, they are also altered in the context of initially well-posed problems: in that case, a minimal demand is that the problem does remain well-posed. This is not trivial, as shown in Section 3.1, and involves a new demand for the regularising operator which implies some coerciveness, whatever the smoothed variable.

Then, a second question of some importance is addressed: do initially ill-posed problems indeed become well-posed once regularised? It is shown in Section 3.5 that the answer depends in general on the range of the regularising operator. Unfortunately, when dealing with operators as designed in Section 3.4 to ensure coerciveness, the problem remains ill-posed, a severe short-coming for this class of regularisation methods.

3.1. Effect on well-posed problems through an example

To illustrate the kind of issues that may be encountered, consider the problem of equilibrium of a clamped bar with some imposed inelastic strain (1D problem) obeying a non-local linear elastic law, based on the gradient penalty operator $\mathbf{P}$ as defined in (21) and Appendix A. Denoting by $e_0(x)$ the imposed inelastic strain field, the energy of the bar reads:

$$W(\varepsilon) = \int_0^l \frac{1}{2} E(\mathbf{P}\varepsilon - e_0)^2 \tag{23}$$

where $\varepsilon = \partial u/\partial x$ is the local strain, u the displacement field, l the bar length and E its stiffness. For the prescribed boundary conditions $u(0) = u(l) = 0$, the problem can be expressed as:

$$\min_{\varepsilon\in KA} W(\varepsilon) \quad \text{with } KA = \{\varepsilon \in L^2(]0,l[);\ \langle\varepsilon\rangle = 0\} \text{ and } \langle\varepsilon\rangle \underset{def.}{=} = \frac{1}{l}\int_0^l \varepsilon\,\mathrm{d}x \tag{24}$$

Although the non-regularised problem (**P** replaced by **Id**) is obviously well-posed, with solution $\varepsilon(x) = e_0(x) - \langle e_0\rangle$, Eq. (24) does not admit solutions for any $e_0(x)$. Indeed, the first variation condition characterising the solution ε is:

$$\forall\delta\varepsilon \in KA \left\langle \frac{\partial W}{\partial\varepsilon}\middle|\delta\varepsilon\right\rangle = \int_0^l \sigma\cdot(\mathbf{P}\,\delta\varepsilon) = \int_0^l (\mathbf{P}^{\mathrm{T}}\sigma)\,\delta\varepsilon = 0 \quad \text{with } \sigma = E(\mathbf{P}\varepsilon - e_0) \tag{25}$$

where $\mathbf{P}^{\mathrm{T}}$ denotes the transpose of **P**. This variational formulation, which expresses the equilibrium of the bar, implies that $\mathbf{P}^{\mathrm{T}}\sigma$ is constant, say equal to Σ. As **P** is self-adjoint (Appendix A, Proposition 1d), $\mathbf{P}^{\mathrm{T}}\sigma = \mathbf{P}\sigma$. Moreover, the inverse image of a constant field is itself (Appendix A, Proposition 1f), so that $\sigma(x) = \Sigma$. The constitutive law then reads:

$$\forall x \quad (\mathbf{P}\varepsilon)(x) = \frac{\Sigma}{E} + e_0(x) \tag{26}$$

As $\mathbf{P}\varepsilon$ should belong to $H^1(]0,l[)$ (Appendix A, Proposition 1), fulfilment of (26) is impossible as soon as the inelastic strain e_0 does not belong itself to $H^1(]0,l[)$, thus precluding the existence of solutions. This is of course achieved with a discontinuous field e_0 and even with continuous distributions of e_0, for instance:

$$e_0(x) = \varepsilon_0\left[1. + \sqrt{\frac{x}{l}}\right] \tag{27}$$

3.2. Mathematical analysis

The previous example shows that, even for very simple structures, the use of a smoothed variable may preclude the existence of solutions, although the problem without regularisation would have been well-posed (linear elasticity). The explanation for this issue is related to the alteration of the properties of the energy function when introducing the regularising operator and requires a mathematical study of the minimisation problem (24).

The energy function W can be expressed as a composition of the regularising operator and the energy for local elasticity Q (quadratic form which is coercive, continuous and strictly convex):

$$\begin{aligned} W = Q\circ\mathbf{P} : L^2 &\xrightarrow{\mathbf{P}} L^2 \xrightarrow{Q} IR \\ \varepsilon \mapsto \bar{\varepsilon} = \mathbf{P}\varepsilon &\mapsto \int_\Omega \frac{1}{2}E(\bar{\varepsilon} - e_0)^2 \end{aligned} \tag{28}$$

Minimisation of W admits solutions if W is proper (not infinite everywhere), lower semi-continuous for L^2 weak topology and coercive (Ekeland and Temam, 1974). Let us confront W with these demands

- W is proper since $W(0) < +\infty$.
- **P** and Q are continuous as soon as e_0 belongs to L^2: W is continuous too and therefore lower semi-continuous for L^2 strong topology. Moreover, **P** is linear and Q is convex: therefore, W is convex. It ensures that W is also lower semi-continuous for L^2 weak topology.
- As $Q(\bar{\varepsilon}) \to \infty \Longleftrightarrow \|\bar{\varepsilon}\| \to \infty$, W is coercive if and only if **P** is coercive in the sense that $\|\varepsilon\| \to \infty \Rightarrow \|\mathbf{P}\varepsilon\| \to \infty$. But **P** *is not coercive* (Appendix A, Proposition 3); neither is W.

In conclusion, the lack of coerciveness for the regularising operator results in the loss of existence of solutions. Note that this problem is not specific to the gradient penalty operator: convolution operators

with a bounded weighting function like (19) are not coercive either (Appendix A, Proposition 3). This fact has already been observed in the context of non-local elasticity (Polizzotto, 2001): to ensure existence of solution, the load fields have to be more regular than for local elasticity.

3.3. Numerical consequences

The lack of solutions has evidently straightforward consequences on the numerical simulation of such problems, as highlighted by a spatial discretisation based on Fourier decomposition. In that way, consider again the clamped bar and assume a discontinuous inelastic strain field e_0:

$$e_0(x) = \varepsilon_0 H\left(\frac{l}{2} - x\right) \quad \text{with } H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0 \end{cases} \tag{29}$$

The numerical solution is based on a usual displacement formulation:

$$\min_{v \in H_0^1(]0,L[)} W\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right) \tag{30}$$

The displacement fields v are approximated by means of N sine functions as interpolation fields:

$$v(x) = \sum_{i=1}^{N} v_i \phi_i(x) \quad \text{where } \phi_i(x) = \sin\frac{i\pi x}{l} \tag{31}$$

Minimisation with respect to displacement fields that belong to this approximation space leads to the following linear diagonal system:

$$K_{ii} u_i = F_i \quad \text{with } K_{ii} = \frac{El}{2}\left(\frac{\frac{i\pi}{l}}{1 + \frac{i^2\pi^2 L_c^2}{l^2}}\right)^2 \text{ and } F_i = E\varepsilon_0 \frac{\sin\left(\frac{i\pi}{2}\right)}{1 + \frac{i^2\pi^2 L_c^2}{l^2}} \tag{32}$$

The closed-form solution of this problem for N degrees of freedom is therefore:

$$u(x) = \sum_{i=1}^{N} u_i \phi_i(x) \quad \text{with } u_i = \begin{cases} 0 & i \text{ even} \\ 2l\,\varepsilon_0 \dfrac{(-1)^{(i+3)/2}}{i^2\pi^2}\left(1 + \dfrac{i^2\pi^2 L_c^2}{l^2}\right) & i \text{ odd} \end{cases} \tag{33}$$

Convergence or not of the solution when the number of degrees of freedom increases is apparent by observing the behaviour of u_i when i grows:

$$\begin{cases} u_i \approx 2l\varepsilon_0 (-1)^{(i+3)/2} \dfrac{L_c^2}{l^2} & \text{if } L_c \neq 0 \\ u_i \approx 2l\varepsilon_0 \dfrac{(-1)^{(i+3)/2}}{i^2\pi^2} & \text{if } L_c = 0 \end{cases} \tag{34}$$

The numerical approximation of the solution converges when the internal length L_c is equal to zero (no regularisation) whereas it does not as soon as $L_c \neq 0$ (regularisation). This is illustrated in Fig. 1: non-controlled oscillations and blowing up of the solution can be observed, mimicking numerical behaviours for more complex problems. Yet, it should be noticed that blowing up of the solution is unmistakable only for high number of modes whereas for coarse discretisations, numerics may almost be believed working, since spurious oscillations are localised in the neighbourhood of the discontinuity only (middle of the bar).

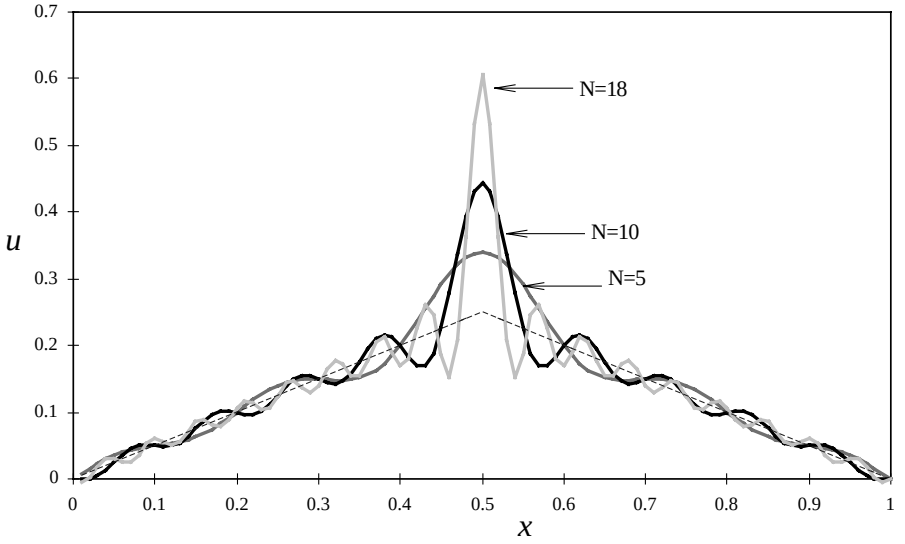

Fig. 1. Clamped bar with discontinuous inelastic strain. Regularisation with operator $\mathbf{P}$ ($L_c = 0.1$).

3.4. A proposal to retrieve coerciveness: a new family of regularising operators

From the previous part, it appears that classical regularising operators fail at preserving the well-posed character of a problem because they are not coercive. Therefore, these operators are inadequate and a new family of regularising operators has to be introduced.

Inspired by inverse problem methods (Tikhonov and Arsenine, 1977) as well as non-local elasticity (Eringen, 1987), we propose the following class of operators:

$$\mathbf{P}_\gamma = \gamma \mathbf{Id} + (1-\gamma)\mathbf{P}, \quad 0 < \gamma < 1 \tag{35}$$

$\gamma = 0$ would correspond to operator $\mathbf{P}$ alone, while $\gamma = 1$ would result in no regularisation at all. Such an operator enjoys all the desirable properties of $\mathbf{P}$ and moreover is coercive, thanks to the identity contribution (Appendix A, Proposition 4).

Numerical evidences of the restored coerciveness when $\gamma \neq 0$ is illustrated again by the numerical treatment of the bar with discontinuous inelastic strain. The closed-form solution for N degrees of freedom of the minimisation problem (30) which now involves the regularising operator $\mathbf{P}_\gamma$ is:

$$u^{(\gamma)}(x) = \sum_{i=1}^{N} u_i^{(\gamma)} \phi_i(x) \quad \text{with } u_i^{(\gamma)} = \begin{cases} 0 & i \text{ even} \\ 2l\varepsilon_0 \dfrac{(-1)^{(i+3)/2}}{i^2\pi^2} \left(\dfrac{1 + \dfrac{i^2\pi^2 L_c^2}{l^2}}{1 + \gamma \dfrac{i^2\pi^2 L_c^2}{l^2}} \right) & i \text{ odd} \end{cases} \tag{36}$$

The behaviour of $u_i^{(\gamma)}$ when i grows is:

$$u_i^{(\gamma)} \approx 2l\varepsilon_0 \frac{(-1)^{(i+3)/2}}{i^2\pi^2} \frac{1}{\gamma} \tag{37}$$

Therefore, the numerical solution converges for any strictly positive value of γ, as illustrated in Fig. 2. Note that the asymptotic convergence speed is independent of the internal length L_c of $\mathbf{P}$.

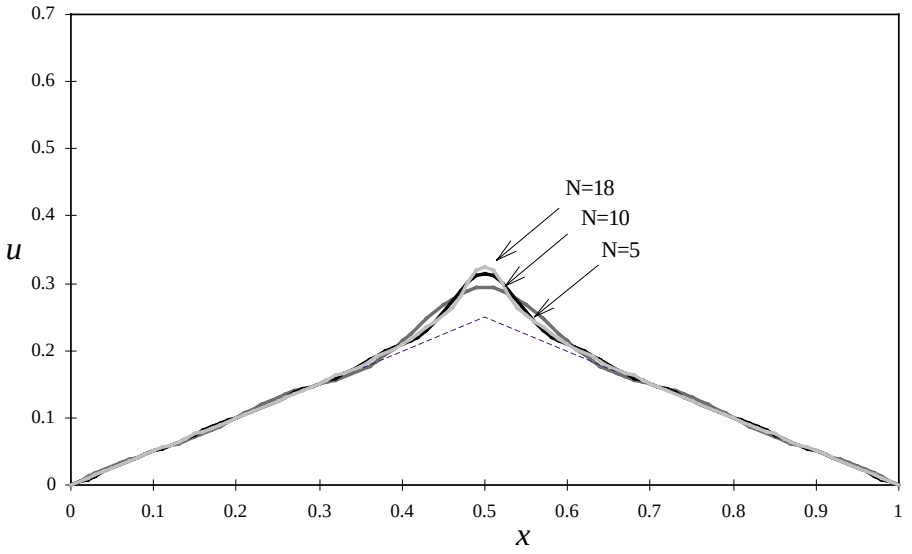

Fig. 2. Clamped bar with discontinuous inelastic strain. Regularisation with operator $\mathbf{P}_\gamma$ ($L_c = 0.1$, $\gamma = 0.2$).

3.5. Effect on ill-posed problems

Replacing a state variable (strain or internal variable) by its smoothed counterpart through an operator $\mathbf{R}$ is actually equivalent to change the space which the variable belongs to, as it can be noticed at once (for the strain, for instance):

$$\underset{\substack{\boldsymbol{\varepsilon}\in V\\ a}}{\arg\min} E(\mathbf{R}\boldsymbol{\varepsilon}, a) = \underset{\substack{\bar{\boldsymbol{\varepsilon}}\in V_R\\ a}}{\arg\min} E(\bar{\boldsymbol{\varepsilon}}, a) \tag{38}$$

Therefore, the question of regularisation is totally included in the nature of the range $V_R = \mathbf{R}(V)$. Namely, if small wavelength functions are included in V_R, then localisation is allowed with a length scale as small as desired for $\bar{\boldsymbol{\varepsilon}}$ and then for $\boldsymbol{\varepsilon}$: there would not be any control of the localisation by the introduction of such a regularising operator. Unfortunately, the usual regularising operators attenuate short wavelengths but do not cut them off. Consequently, introducing regularisation through such operators does not prevent from localisation: initially ill-posed problems remain ill-posed.

To illustrate this disappointing observation, consider again the example of the clamped bar. To ensure coerciveness, we choose the regularising operator $\mathbf{P}_\gamma$ (35). $\mathbf{P}_\gamma$ as well as $\mathbf{P}$ preserves the mean value of a field (straightforward application of Appendix A, Proposition 1e) and is bijective (Appendix A, Proposition 5), so that:

$$\begin{cases} \mathbf{P}_\gamma(L^2(]0,l[)) = L^2(]0,l[) \\ \langle \mathbf{P}_\gamma \varepsilon \rangle = \langle \varepsilon \rangle \end{cases} \Rightarrow \mathbf{P}_\gamma(KA) = KA \tag{39}$$

where we remind that KA denotes the admissible strain for the clamped bar problem, as defined in (24). Therefore, there is not any regularisation of the set of kinematically admissible strain fields ($V_R = V$). (38) then allows to conclude that the regularised problem is as ill-posed as the initial local one may have been: the regularisation is not effective. This constitutes a severe breakpoint for such approaches based on coupled regularisation.

4. Split regularisation

The failure of the previous class of models to deal with strain softening is related to the fact that a variable is replaced by its smoothed counterpart in the whole expression of the energy, so that its only effect is to change the function space within which the variable is sought. Therefore, a more promising approach consists in mixing local and smoothed variables. A possible way consists in keeping the local internal variables within the expression of the dissipation potential while replacing them by their smoothed counterparts into the expression of the free energy, second line of Table 1:

$$(\boldsymbol{\varepsilon}^*, a^*) = \arg\min_{\boldsymbol{\varepsilon},a} E_R(\boldsymbol{\varepsilon}, a) \quad \text{with } E_R(\boldsymbol{\varepsilon}, a) = F_R(\boldsymbol{\varepsilon}, a) + D(a - a^-) \text{ and } F_R(\boldsymbol{\varepsilon}, a) = F(\boldsymbol{\varepsilon}, \mathbf{R}a) \tag{40}$$

In that way, the local evolution equations, governed by the dissipation potential, are preserved while the state equations, governed by the free energy, are altered by the regularising operator $\mathbf{R}$, hence a possible link with driving force regularisation as presented in the literature (see Pijaudier-Cabot and Bazant, 1987; Comi and Perego, 2001). This link is explored below through the example of isotropic plasticity.

This class of models based on split regularisation is examined in terms of effect on well-posed problems as well as effective regularisation. Unfortunately, it leads to the same conclusions as previously:

- a coercive operator is generally required for the regularisation,
- regularisation does not improve the behaviour of the model with respect to localisation phenomena.

This is in apparent contradiction with the literature which asserts that driving force regularisation does lead to effective regularisation. This paradox will be resolved in Section 6 for a specific constitutive law.

4.1. Expression of the non-local constitutive relation

First, as the strain field is not altered, the stress keeps its usual definition, see (17):

$$\forall \delta\boldsymbol{\varepsilon} \in KA^0 \quad \left\langle \frac{\partial F_R}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, a) \middle| \delta\boldsymbol{\varepsilon} \right\rangle = \int_\Omega \boldsymbol{\sigma}(x) \cdot \delta\boldsymbol{\varepsilon}(x) = 0 \quad \text{with } \boldsymbol{\sigma} = \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \mathbf{R}a) \tag{41}$$

Moreover, a characterisation of the minimum (40) with respect to a is given by (13) and (14) when the time step goes to zero and provides the evolution equation:

$$-\frac{\partial F_R}{\partial a}(\boldsymbol{\varepsilon}, a) \in \partial D(\dot{\boldsymbol{a}}) \tag{42}$$

It involves the global driving force whose expression is:

$$\forall \delta a \quad \left\langle \frac{\partial F_R}{\partial a}(\boldsymbol{\varepsilon}, a) \middle| \delta a \right\rangle = \int_\Omega \frac{\partial \Phi}{\partial a}(\boldsymbol{\varepsilon}, \mathbf{R}a)(\mathbf{R}\delta a) = \int_\Omega \mathbf{R}^{\mathrm{T}} \left(\frac{\partial \Phi}{\partial a}(\boldsymbol{\varepsilon}, \mathbf{R}a) \right) \delta a \tag{43}$$

By taking (42) and (43) into account, the evolution law associated with the regularised incremental potential energy simply reduces to the following pointwise law:

$$\overline{A} \in \partial \Delta(\dot{\boldsymbol{a}}) \quad \text{with } \overline{A} = -\mathbf{R}^{\mathrm{T}} \frac{\partial \Phi}{\partial a}(\boldsymbol{\varepsilon}, \mathbf{R}a) \tag{44}$$

Or equivalently:

$$\begin{cases} \dot{\boldsymbol{a}} = \lambda \dfrac{\partial f}{\partial A}(\overline{A}) \\ f(\overline{A}) \leqslant 0, \ \lambda \geqslant 0, \ \lambda f(\overline{A}) = 0 \end{cases} \quad \text{with } \overline{A} = -\mathbf{R}^{\mathrm{T}} \frac{\partial \Phi}{\partial a}(\boldsymbol{\varepsilon}, \mathbf{R}a) \tag{45}$$

4.2. Link with driving force regularisation

Eq. (45) shows that the evolution of the internal variable is governed by the same evolution equation as the local model but with two distinct features:

- the driving force appearing in this equation is the original force, smoothed by the transpose of the regularising operator defined for the internal variable field;
- the original force has to be evaluated with the smoothed internal variable.

Comparison between this regularised energy approach and other propositions of the literature can be achieved by looking at a simple model of plasticity. This model is described by three state variables $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{\mathrm{p}}, p)$: the strain, the plastic strain and a hardening variable. The local free Helmholtz' energy, the stress and the driving forces associated to the plastic strain and the hardening variable are:

$$\begin{aligned} &\Phi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{\mathrm{p}}, p) = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}) \cdot \mathbf{E} \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}) + \frac{1}{2} h p^2 \\ &\begin{cases} \boldsymbol{\sigma} = \dfrac{\partial \Phi}{\partial \boldsymbol{\varepsilon}} = \mathbf{E} \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}) \\ \mathbf{A}_{\boldsymbol{\varepsilon}^{\mathrm{p}}} = -\dfrac{\partial \Phi}{\partial \boldsymbol{\varepsilon}^{\mathrm{p}}} = \mathbf{E} \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}) = \boldsymbol{\sigma} \\ A_p = -\dfrac{\partial \Phi}{\partial p} = -hp \end{cases} \end{aligned} \tag{46}$$

where $\mathbf{E}$ is the elasticity tensor and h is the hardening (positive) or softening (negative) modulus. The dissipation potential is defined as in (4) by a yield function f which depends on the driving forces only, so that its subgradient is characterised by a consistency condition and normal flow rules:

$$\begin{cases} f(\boldsymbol{\sigma}, A_p) \leqslant 0, \;\; \lambda \geqslant 0, \;\; \lambda f(\boldsymbol{\sigma}, A_p) = 0 \\ \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \lambda \dfrac{\partial f}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, A_p), \;\; \dot{p} = \lambda \dfrac{\partial f}{\partial A_p}(\boldsymbol{\sigma}, A_p) \end{cases} \tag{47}$$

A model with smoothed force as understand from the related literature can be built in a simple way by substituting a smoothed driving force for A_p into the evolution equations (47), therefore named driving force regularisation. Denoting by $\mathbf{r}$ the applied linear regularising operator, the evolution equation becomes:

$$\begin{cases} f(\boldsymbol{\sigma}, \overline{A}) \leqslant 0, \;\; \lambda \geqslant 0, \;\; \lambda f(\boldsymbol{\sigma}, \overline{A}) = 0 \\ \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \lambda \dfrac{\partial f}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \overline{A}), \;\; \dot{p} = \lambda \dfrac{\partial f}{\partial A_p}(\boldsymbol{\sigma}, \overline{A}) \end{cases} \quad \text{with } \overline{A} = \mathbf{r} A_p(p) \tag{48}$$

This equation is compared to the evolution equation obtained by the regularised energy approach (45) with a regularising operator $\mathbf{R}$ acting only [3] on the internal variable p:

$$\begin{cases} f(\boldsymbol{\sigma}, \overline{A}) \leqslant 0, \;\; \lambda \geqslant 0, \;\; \lambda f(\sigma, \overline{A}) = 0 \\ \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \lambda \dfrac{\partial f}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \overline{A}), \;\; \dot{p} = \lambda \dfrac{\partial f}{\partial A_p}(\boldsymbol{\sigma}, \overline{A}) \end{cases} \quad \text{with } \overline{A} = \mathbf{R}^{\mathrm{T}} A_p(\mathbf{R}p) \tag{49}$$

[3] This is not in agreement with the assumption that all the internal variables are smoothed with the same operator. Nevertheless, this is not of great importance regarding the question of existence of solution that will be examined in the next section.

When taking into account the expression of A_p given in (46), both models coincide in this simple case if:

$$\begin{aligned} \mathbf{r}A_p(p) = \mathbf{R}^{\mathrm{T}}A_p(\mathbf{R}p) &\Longleftrightarrow -h\mathbf{r}p = -h\mathbf{R}^{\mathrm{T}}\mathbf{R}p \\ &\Updownarrow \\ \mathbf{r} &= \mathbf{R}^{\mathrm{T}}\mathbf{R} \end{aligned} \tag{50}$$

This example shows the link between driving force regularisation and split regularisation as soon as the regularising operator $\mathbf{r}$ acting on the driving force is self-adjoint and positive. Of course, the exact coincidence in this case results from the linearity of A_p with respect to p. Generally, both classes of models are only strongly related, through a comparison between (45) and (48).

4.3. A demand for coerciveness

As the evolution equation is expressed as a minimisation with respect to the internal variables (40), the question of coerciveness should be raised again. We could have expected that the dissipation potential D which depends on local variables may ensure coerciveness. Unfortunately, for rate-independent materials, D is positive homogeneous of degree one only, so that the coerciveness of E in appropriate function spaces is *generally* brought through the free energy F (see Section 6.2 for an exception). Therefore, the arguments in Section 3.2 are still valid: the regularising operator $\mathbf{R}$ has again to be coercive.

The following example which is inspired of (Engelen, 1999) illustrates this assertion by showing once more that a non-coercive regularising operator may preclude the existence of solution. Consider a 1D bar of length l whose constitutive law is the non-local strain-hardening ($h > 0$) plasticity model described previously with von Mises yield function:

$$f(\sigma, \overline{A}) = \sigma_{\mathrm{eq}} + \overline{A} - \sigma^{\mathrm{y}} \quad \text{with } \overline{A} = \mathbf{P}^{\mathrm{T}}(-h\mathbf{P}p) = -h\mathbf{P}^2 p \tag{51}$$

where σ_{eq} is von Mises' effective stress and σ^{y} is the yield stress, for which a spatial distribution of the following form is assumed:

$$\sigma^{\mathrm{y}} \notin H^1(]0l[) \quad \text{for instance} \begin{cases} \sigma^{\mathrm{y}} \text{ discontinuous} \\ \text{or} \\ \sigma^{\mathrm{y}}(x) = \sigma_0\left[1 + \sqrt{x/l}\right] \end{cases} \tag{52}$$

For high enough loading, plasticity occurs everywhere, so that the consistency condition leads to:

$$\forall x \quad f(\sigma, \overline{A}) = 0 \Longleftrightarrow (\mathbf{P}^2 p)(x) = \frac{\sigma_{\mathrm{eq}} - \sigma^{\mathrm{y}}(x)}{h} \tag{53}$$

Thanks to the equilibrium equation, σ_{eq} is constant; thus, the right hand side does not belong to $H^1(\Omega)$ while the left hand side should: no solution exists although the local problem obviously admits one (obtained by replacing $\mathbf{P}$ by $\mathbf{Id}$).

4.4. Localisation analysis

In the case of split regularisation, the local internal variable and its smoothed counterpart are simultaneously present in the expression of the energy, so that we could hope that the effective regularisation property is enhanced compared to coupled regularisation (Section 3.5). Unfortunately, a localisation analysis as described in Appendix B leads to the opposite result, as shown below.

The analysis relies on the determination of the non-local model tangent operator for a homogeneous material state $(\boldsymbol{\varepsilon}^0, a^0)$. First, differentiation of the stress–strain relation (43) with respect to time along with the flow rule (47) lead to:

$$\dot{\boldsymbol{\sigma}} = \frac{\partial^2 \boldsymbol{\Phi}}{\partial \boldsymbol{\varepsilon}\, \partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^0, a^0)\dot{\boldsymbol{\varepsilon}} + \frac{\partial^2 \boldsymbol{\Phi}}{\partial \boldsymbol{\varepsilon}\, \partial a}(\boldsymbol{\varepsilon}^0, a^0)\mathbf{R}\dot{\boldsymbol{a}} \quad \text{with } \dot{\boldsymbol{a}} = \lambda \frac{\partial f}{\partial A}(A^0) \tag{54}$$

where the homogeneous character of the material state has been explicitly used, along with the fact that constant fields are left unchanged by the regularising operator. Then, thanks to the assumption of linear comparison solid, the consistency condition reduces to:

$$\dot{f} = 0 \Rightarrow \frac{\partial f}{\partial A}(A^0)\mathbf{R}^{\mathrm{T}}\left[\frac{\partial^2 \Phi}{\partial \boldsymbol{\varepsilon}\,\partial a}(\boldsymbol{\varepsilon}^0, a^0)\dot{\boldsymbol{\varepsilon}} + \frac{\partial^2 \Phi}{\partial a^2}(\boldsymbol{\varepsilon}^0, a^0)\mathbf{R}\left(\lambda \frac{\partial f}{\partial A}(A^0)\right)\right] = 0 \tag{55}$$

Now, we recall that the *same* linear regularising operator is applied on all the internal variables, an essential requirement here. Then (55) is equivalent to:

$$\mathbf{R}^{\mathrm{T}}\left[\frac{\partial f}{\partial A}(A^0)\frac{\partial^2 \Phi}{\partial \boldsymbol{\varepsilon}\,\partial a}(\boldsymbol{\varepsilon}^0, a^0)\dot{\boldsymbol{\varepsilon}} + \frac{\partial f}{\partial A}(A^0)\frac{\partial^2 \Phi}{\partial a^2}(\boldsymbol{\varepsilon}^0, a^0)\frac{\partial f}{\partial A}(A^0)\mathbf{R}\lambda\right] = 0 \tag{56}$$

If $\mathbf{R}^{\mathrm{T}}$ is assumed injective (obviously true for self-adjoint injective operators), Eq. (56) allows to derive $\mathbf{R}\lambda$ since it is equivalent to:

$$\frac{\partial f}{\partial A}(A^0)\frac{\partial^2 \Phi}{\partial a\,\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^0, a^0)\dot{\boldsymbol{\varepsilon}} + \left[\frac{\partial f}{\partial A}(A^0)\frac{\partial^2 \Phi}{\partial a^2}(\boldsymbol{\varepsilon}^0, a^0)\frac{\partial f}{\partial A}(A^0)\right]\mathbf{R}\lambda = 0 \tag{57}$$

Finally, the rate constitutive law reads:

$$\dot{\boldsymbol{\sigma}} = \left[\frac{\partial^2 \Phi}{\partial \boldsymbol{\varepsilon}\,\partial \boldsymbol{\varepsilon}} + \frac{1}{\dfrac{\partial f}{\partial A}\dfrac{\partial^2 \Phi}{\partial a\,\partial a}\dfrac{\partial f}{\partial A}}\left(\frac{\partial^2 \Phi}{\partial \boldsymbol{\varepsilon}\,\partial a}\frac{\partial f}{\partial A}\right) \otimes \left(\frac{\partial f}{\partial A}\frac{\partial^2 \Phi}{\partial a\,\partial \boldsymbol{\varepsilon}}\right)\right] \cdot \dot{\boldsymbol{\varepsilon}} = \mathbf{H} \cdot \dot{\boldsymbol{\varepsilon}} \tag{58}$$

where $\mathbf{H}$ is actually the tangent operator for the local constitutive relation, as defined in (B.1). In consequence, the rate constitutive law for the regularised model is the same as for the purely local model: $\mathbf{R}$ does not bring much regularisation. The localisation analysis in Appendix B explains why the same pathologies are to be expected for the non-local model and the local one.

5. Uncoupled regularisation

At last, we analyse the effect of smoothing one variable (strain or set of internal variables) and minimising with respect to the other (respectively set of internal variables or strain), third line of Table 1. It corresponds to the following non-local models:

strain regularisation:

$$\begin{cases} \boldsymbol{\varepsilon}^* = \underset{\boldsymbol{\varepsilon}}{\arg\min}\, E(\boldsymbol{\varepsilon}, a^*) \\ a^* = \underset{a}{\arg\min}\, E(\mathbf{R}\boldsymbol{\varepsilon}^*, a) \end{cases} \underset{\Delta t \to 0}{\longrightarrow} \begin{cases} \boldsymbol{\sigma} = \dfrac{\partial \Phi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, a) \\ A = -\dfrac{\partial \Phi}{\partial a}(\mathbf{R}\boldsymbol{\varepsilon}, a) \in \partial \Delta(\dot{a}) \end{cases} \tag{59}$$

internal variable regularisation:

$$\begin{cases} \boldsymbol{\varepsilon}^* = \underset{\boldsymbol{\varepsilon}}{\arg\min}\, E(\boldsymbol{\varepsilon}, \mathbf{R}a^*) \\ a^* = \underset{a}{\arg\min} E(\boldsymbol{\varepsilon}^*, a) \end{cases} \underset{\Delta t \to 0}{\longrightarrow} \begin{cases} \boldsymbol{\sigma} = \dfrac{\partial \Phi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \mathbf{R}a) \\ A = -\dfrac{\partial \Phi}{\partial a}(\boldsymbol{\varepsilon}, a) \in \partial \Delta(\dot{a}) \end{cases} \tag{60}$$

Expression (59) allows to recognize a close relation between strain regularisation and some proposals of the literature (namely Pijaudier-Cabot, 1995; Peerlings et al., 1996; Comi, 2001), even though these authors do

not smooth the whole strain tensor but only some of its invariants. Internal variable regularisation (60) encompasses non-local damage as proposed by Bazant and Pijaudier-Cabot (1988), except for their loading/unloading condition which depends on the non-local damage variable to enforce the pointwise Clausius–Duhem inequality whereas the consistency condition is expressed in terms of the local damage variable.

In these proposals of the literature, the coerciveness of the regularising operator has not been questioned. Actually, it does not mind. Indeed, the coerciveness of $\mathbf{R}$ does not play a role anymore since the minimisation is not led with respect to the smoothed variable. Therefore, regularising operators as convolution ones or gradient penalty $\mathbf{P}$ can be applied without special care.

However, the problem cannot be expressed as a single minimisation problem with respect to both the strain and the internal variables since the energies used in the equilibrium equation and the constitutive equation are not the same anymore. That is why this approach may appear as an algorithmic scheme rather than a thermodynamics-based formulation. In particular, the global dissipation is not prevented a priori from being negative.

Despite this drawback, one can expect to gain finally some effective regularisation as pointed out in the literature. It is confirmed by a localisation analysis which leads to tangent operators that exhibit a stabilising term compared to the local model, unexpectedly the same one for strain regularisation as for internal variable regularisation.

5.1. Thermodynamic consequences of the loss of a single energy

By uncoupling minimisation and regularisation, the definition of a single energy used in both equilibrium equation (minimisation with respect to the strain) and constitutive equation (minimisation with respect to the internal variables) is lost. On a mathematical ground, it results generally in a non-symmetric formulation, while on a physical ground, the Clausius–Duhem inequality is not automatically fulfilled. This is a departure from the framework of generalised standard materials.

Indeed, consider the non-local constitutive equations (59) or (60) obtained when the time increment Δt goes to zero. The usual equilibrium equations are preserved, which means that the stress $\boldsymbol{\sigma}$ that appears in the constitutive relation (59) or (60) also respects equilibrium. To express the dissipation, the free energy has to be defined. This is achieved by the usual argument that for reversible load history, the dissipation is equal to zero, so that the stress can be expressed as the variation of the free energy with respect to the strain. This leads to the following choices of free energy and expressions for the dissipation:

strain regularisation:

$$F(\boldsymbol{\varepsilon}, a) = \int_{\Omega} \Phi(\boldsymbol{\varepsilon}, a) \Rightarrow D = \int_{\Omega} \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}} - \dot{F} = \int_{\Omega} \left[-\frac{\partial \Phi}{\partial a}(\boldsymbol{\varepsilon}, a) \right] \dot{a} \tag{61}$$

internal variable regularisation:

$$F(\boldsymbol{\varepsilon}, a) = \int_{\Omega} \Phi(\boldsymbol{\varepsilon}, \mathbf{R}a) \Rightarrow D = \int_{\Omega} \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}} - \dot{F} = \int_{\Omega} \left[-\mathbf{R}^{\mathrm{T}} \frac{\partial \Phi}{\partial a}(\boldsymbol{\varepsilon}, \mathbf{R}a) \right] \dot{a} \tag{62}$$

Unfortunately, the constitutive relation (59) or (60) only ensures that:

$$\forall x \in \Omega \quad A(x)\dot{a}(x) \geqslant 0 \tag{63}$$

where A is no more equal to the dual variable of $\dot{a}$ within the expression of the dissipation. Therefore, the dissipation cannot be claimed positive a priori.

5.2. Localisation analysis for strain regularisation

To examine the regularisation properties of this class of models, we rely on the results given in Appendix B. The tangent operator of the model has to be derived for a homogeneous state $(\boldsymbol{\varepsilon}^0, a^0)$. Starting with (59), straightforward calculations show that:

$$\dot{\boldsymbol{\sigma}} = \frac{\partial^2 \Phi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^0, a^0) \cdot \dot{\boldsymbol{\varepsilon}} + \frac{\left(\dfrac{\partial^2 \Phi}{\partial \boldsymbol{\varepsilon} \partial a}(\boldsymbol{\varepsilon}^0, a^0) \dfrac{\partial f}{\partial A}(A)\right) \otimes \left(\dfrac{\partial f}{\partial A}(A) \dfrac{\partial^2 \Phi}{\partial a \partial \boldsymbol{\varepsilon}}(\mathbf{R}\boldsymbol{\varepsilon}^0, a^0)\right)}{\dfrac{\partial f}{\partial A}(A) \dfrac{\partial^2 \Phi}{\partial a \partial a}(\mathbf{R}\boldsymbol{\varepsilon}^0, a^0) \dfrac{\partial f}{\partial A}(A)} \cdot \mathbf{R}\dot{\boldsymbol{\varepsilon}} \tag{64}$$

Taking advantage of the fact that $\mathbf{R}$ leaves the homogeneous field $\boldsymbol{\varepsilon}^0$ unchanged, and introducing the tangent operator for the local model $\mathbf{H}$ and its elastic part $\mathbf{E}$, as defined in (B.2), one finally gets:

$$\dot{\boldsymbol{\sigma}} = \mathbf{H} \cdot \mathbf{R}\dot{\boldsymbol{\varepsilon}} + \mathbf{E} \cdot (\dot{\boldsymbol{\varepsilon}} - \mathbf{R}\dot{\boldsymbol{\varepsilon}}) \tag{65}$$

Appendix B then allows to conclude. Thanks to its definite positiveness, $\mathbf{E}$ brings some stability in (65). The bifurcation analysis shows that a non-zero minimal wavelength is selected with a non-coercive operator $\mathbf{R}$, whatever the softening modulus. Therefore, the width of the localisation band cannot be zero: there is indeed regularisation compared to the local model. On the contrary, with coercive operator $\mathbf{R}$, the minimal wavelength depends on the softening modulus and may vanish for too steep softening. That is why non-coercive regularising operators should be preferred in this approach.

5.3. Localisation analysis for internal variable regularisation

A localisation analysis is performed in the same way as for strain regularisation. After straightforward calculations, the tangent operator of the non-local model for a homogeneous material state (ε^0, a^0) reads:

$$\dot{\boldsymbol{\sigma}} = \frac{\partial^2 \Phi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^0, \mathbf{R}a^0) \cdot \dot{\boldsymbol{\varepsilon}} + \frac{\partial^2 \Phi}{\partial \boldsymbol{\varepsilon} \partial a}(\boldsymbol{\varepsilon}^0, \mathbf{R}a^0)\mathbf{R} \left[\frac{\dfrac{\partial f}{\partial A}(A^0) \dfrac{\partial f}{\partial A}(A^0) \dfrac{\partial^2 \Phi}{\partial a \partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^0, a^0) \cdot \dot{\boldsymbol{\varepsilon}}}{\dfrac{\partial f}{\partial A}(A^0) \dfrac{\partial^2 \Phi}{\partial a \partial a}(\boldsymbol{\varepsilon}^0, a^0) \dfrac{\partial f}{\partial A}(A^0)} \right] \tag{66}$$

Taking advantage again of the fact that $\mathbf{R}$ leaves homogeneous fields unchanged, one finally gets the same expression for the tangent operator as obtained with strain regularisation (65):

$$\dot{\boldsymbol{\sigma}} = \mathbf{H} \cdot \mathbf{R}\dot{\boldsymbol{\varepsilon}} + \mathbf{E} \cdot (\dot{\boldsymbol{\varepsilon}} - \mathbf{R}\dot{\boldsymbol{\varepsilon}}) \tag{67}$$

The conclusions of the localisation analysis are the same: both non-local models bring effective regularisation, preferably with a non-coercive regularising operator $\mathbf{R}$.

6. Application to a brittle damage model

At this stage, the analysis can be summed up as follows. Under the assumptions recalled in Section 2.3, we observe that:

- coupled regularisation requires a coercive regularising operator which actually does not provide any regularisation;
- split regularisation generally requires a coercive regularising operator which does not either provide any regularisation;

- uncoupled regularisation brings some regularisation, preferably with a non-coercive operator. But the thermodynamic framework is lost: the Clausius–Duhem inequality is not automatically fulfilled.

However, it has been noticed in Section 4.2 that split regularisation is close to driving force regularisation, which indeed allows a treatment of strain-softening laws, according to the literature (see Pijaudier-Cabot and Bazant, 1987; Comi and Perego, 2001). This is in apparent contradiction with our analysis.

Actually, the results in the literature are generally obtained with a specific constitutive law, designed to model brittle elasticity. The material state is characterised by the strain $\boldsymbol{\varepsilon}$ and a scalar damage variable d which ranges from 0 (sound material) to 1 (broken one). The stress–strain relation depends linearly on d. The expression is slightly different in the two articles; it respectively reads:

$$\begin{aligned} &\boldsymbol{\sigma} = (1-d)\mathbf{E}^0 \cdot \boldsymbol{\varepsilon} && \text{(a) (Pijaudier-Cabot and Bazant, 1987)} \\ &\boldsymbol{\sigma} = (1-\mathbf{R}d)\mathbf{E}^0 \cdot \boldsymbol{\varepsilon} && \text{(b) (Comi and Perego, 2001)} \end{aligned} \tag{68}$$

with $\mathbf{E}^0$ the initial stiffness. The evolution of damage is governed by a yield function of the smoothed energy density release rate $\overline{Y}$:

$$\dot{d} \geqslant 0,\ \overline{Y} - k(d) \leqslant 0,\ \dot{d}(\overline{Y} - k(d)) = 0 \quad \text{with} \begin{cases} Y = \dfrac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbf{E}^0 \cdot \boldsymbol{\varepsilon} \\ \overline{Y} = \mathbf{R}^{\mathrm{T}} Y \end{cases} \tag{69}$$

The essential difference with (41)–(45) is the dependence of the yield function with respect to the (local) damage variable: the model does not belong to the framework of generalised standard materials.

A localisation analysis shows that both models lead indeed to regularising properties. But the slight difference between (68)a and b results in a different tangent operator, namely:

$$\begin{aligned} &\dot{\boldsymbol{\sigma}} = \mathbf{E} \cdot (\dot{\boldsymbol{\varepsilon}} - \mathbf{R}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}) + \mathbf{H} \cdot \mathbf{R}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}} && \text{(a) (Pijaudier-Cabot and Bazant, 1987)} \\ &\dot{\boldsymbol{\sigma}} = \mathbf{E} \cdot (\dot{\boldsymbol{\varepsilon}} - \mathbf{R}\mathbf{R}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}) + \mathbf{H} \cdot \mathbf{R}\mathbf{R}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}} && \text{(b) (Comi and Perego, 2001)} \end{aligned} \tag{70}$$

We propose now two ways of recasting such models within the framework of generalised standard materials and examine the conclusions of our analysis.

6.1. Single internal variable and uncoupled regularisation

The local constitutive relation corresponding to (68) and (69) does not appear as a generalised standard material because the threshold, thus the dissipation potential, depends explicitly on the internal variable and not only on the driving force. To retrieve nevertheless a generalised standard material, the idea consists in moving the dependence of the threshold from the dissipation potential to the free energy, thus altering the definition of the driving force, and preserving a constant threshold in the dissipation potential. This is achieved by the following specific choice of free energy and yield function, where a kind of stored energy appears in the free energy:

$$\Phi(\boldsymbol{\varepsilon}, d) = \frac{1}{2}(1-d)\boldsymbol{\varepsilon} \cdot \mathbf{E}^0 \cdot \boldsymbol{\varepsilon} + \int_0^d (k(s) - k^0)\,\mathrm{d}s \quad \text{with } k^0 = k(0) \tag{71}$$

$$f(A) = A - k^0 \tag{72}$$

with A the new driving force associated to damage:

$$A = \tfrac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbf{E}^0 \cdot \boldsymbol{\varepsilon} - k(d) + k^0 = Y - k(d) + k^0 \tag{73}$$

Such a model is generalised standard and equivalent to the local version of (68) and (69).

With these expressions, the assumptions of our analysis are fulfilled. Therefore, to get any regularising properties, only an uncoupled regularisation may succeed. By choosing to regularise the strain field through the operator $\mathbf{R}^{\mathrm{T}}$, a straightforward application of (59) leads to:

$$\boldsymbol{\sigma} = (1-d)\mathbf{E}^0 \cdot \boldsymbol{\varepsilon} \tag{74}$$

$$\dot{d} \geqslant 0,\ \widetilde{Y} - k(d) \leqslant 0,\ \dot{d}(\widetilde{Y} - k(d)) = 0 \quad \text{with } \widetilde{Y} = \frac{1}{2}(\mathbf{R}^{\mathrm{T}}\boldsymbol{\varepsilon}) \cdot \mathbf{E}^0 \cdot (\mathbf{R}^{\mathrm{T}}\boldsymbol{\varepsilon}) \tag{75}$$

The difference between (68a) and (69) and (74) and (75) relies in the definition of the smoothed driving force. It can be expected that both are close:

$$\overline{Y} = \mathbf{R}^{\mathrm{T}}\left(\frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbf{E}^0 \cdot \boldsymbol{\varepsilon}\right) \approx \frac{1}{2}(\mathbf{R}^{\mathrm{T}}\boldsymbol{\varepsilon}) \cdot \mathbf{E}^0 \cdot (\mathbf{R}^{\mathrm{T}}\boldsymbol{\varepsilon}) = \widetilde{Y} \tag{76}$$

This closeness is confirmed by the fact that both models lead to the same tangent operator for homogeneous state $(\boldsymbol{\varepsilon}^0, d^0)$, as used in the localisation analysis.

6.2. Two internal variables and split regularisation

In the previous sub-section, a formulation close to that used in the literature is obtained through uncoupled regularisation. Its main drawback consists in the lack of thermodynamic bases. Therefore, we examine here another approach: it is based on split regularisation, so that it respects thermodynamics. However, our analysis showed that such an approach was doomed regarding regularising properties. This conclusion relied on:

- the usual lack of coerciveness brought by the dissipation potential alone, therefore requiring a coercive regularising operator;
- the same regularisation for all internal variables (obviously fulfilled when there is only one) leading to the same tangent operator as for the purely local model.

Therefore, to get an effective regularisation through split regularisation, the model should exhibit at least two internal variables and the corresponding dissipation potential should bring enough coerciveness so that the regularising operator need not be coercive. At last, a specific localisation analysis should be led since no general results are available in that case.

The first step consists in proposing a local model with two internal variables and equivalent to the local model corresponding to (68) and (69). For that purpose, we depart from (71) and (72). The free energy exhibits two terms, the elastic energy and a stored energy, the latter being responsible for the change in the effective threshold. Now, we assume that these mechanisms are described by separate internal variables: d still measures the decrease of the elastic modulus while a new internal variable κ governs the evolution of the effective threshold. The corresponding free energy and the driving force expressions read:

$$\Phi(\boldsymbol{\varepsilon}, d, \kappa) = \frac{1}{2}(1-d)\boldsymbol{\varepsilon} \cdot \mathbf{E}^0 \cdot \boldsymbol{\varepsilon} + \int_0^{\kappa} (k(s) - k^0)\,\mathrm{d}s \tag{77}$$

$$Y = -\frac{\partial \Phi}{\partial d} = \frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbf{E}^0 \cdot \boldsymbol{\varepsilon} \quad \text{and} \quad K = -\frac{\partial \Phi}{\partial \kappa} = k^0 - k(\kappa) \tag{78}$$

Then, to get a model equivalent to (71) and (72), we specialise this more general model by enforcing that both internal variables remain equal to each other by means of the evolution equations. This is achieved by the following dissipation potential:

$$\Delta(\dot{d},\dot{\kappa}) = k^0\dot{d} + \underbrace{I_{IR^+}(\dot{d})}_{\substack{\text{enforces}\\ \dot{d}\geqslant 0}} + \underbrace{I_{\{0\}}(\dot{\kappa}-\dot{d})}_{\substack{\text{enforces}\\ \dot{\kappa}=\dot{d}}} \quad \text{where} \quad I_K(x) = \begin{cases} 0 & \text{if } x\in K \\ +\infty & \text{if } x\notin K \end{cases} \tag{79}$$

The flow rules and the consistency condition are expressed through an interpretation of (5) with the dissipation potential (79):

$$\begin{cases} \dot{\kappa} = \lambda \\ \dot{d} = \lambda \end{cases} \quad \text{and} \quad \lambda \geqslant 0, \quad Y + K - k^0 \leqslant 0, \quad \lambda(Y + K - k^0) = 0 \tag{80}$$

Thus, Eqs. (77)–(80) is equivalent to (71) and (72). The first step is achieved: a generalised standard model with two internal variables and equivalent to the previous one is derived.

The next step consists in regularising this model through a split regularisation. Here, only the damage variable d is smoothed. Thanks to this choice, a straightforward application of (41), (45) and (80) leads to the following constitutive equations, which appears to be exactly the model proposed by Comi and Perego (2001):

$$\boldsymbol{\sigma} = (1 - \mathbf{R}d)\mathbf{E}^0 \cdot \boldsymbol{\varepsilon} \tag{81}$$

$$\dot{d} \geqslant 0,\ \overline{Y} + K - k^0 \leqslant 0,\ \dot{d}(\overline{Y} + K - k^0) = 0 \quad \text{with} \quad \begin{cases} \overline{Y} = \mathbf{R}^{\mathrm{T}} Y = \mathbf{R}^{\mathrm{T}}\left(\dfrac{1}{2}\boldsymbol{\varepsilon}\cdot\mathbf{E}^0\cdot\boldsymbol{\varepsilon}\right) \\ K = k^0 - k(d) \end{cases} \tag{82}$$

Finally, the regularising properties and the coerciveness of the model have to be checked, since some assumptions of our main analysis have been relaxed. The localisation analysis proves the regularising properties, see (70)b for the tangent operator. The coerciveness should be examined with respect to the pair (d,κ). In fact, thanks to the indicator function within the dissipation potential which enforces $d = \kappa$, the energy function is coercive with respect to (d,κ) as soon it is coercive with respect to κ alone. And this latter is ensured thanks to the stored energy term which is not affected by the regularising operator. More precisely, the function space which the internal variables belong to depends on the form of the stored energy, that is on the threshold function $k(d)$.

6.3. Smoothing the hardening variable

In the previous section, we succeed in deriving a non-local model that exhibits both thermodynamic foundations and effective regularisation. However, this is possible only by giving up the assumptions of the general analysis: two different regularising operators have been applied on both internal variables (namely **R** and **Id** respectively on d and κ). But it should be emphasised that this good result is actually highly dependent on the model.

As an illustration, consider again the same local model (78)–(80) with two internal variables, but now, the hardening variable κ is smoothed instead of the damage one d. Again, the dissipation potential alone does not bring enough coerciveness with respect to the pair (d,κ) to proceed in reasonable function spaces: the free energy has still to ensure coerciveness with respect to κ. But now, the stored energy term is affected by the regularisation, so that a coercive regularising operator **R** should be adopted, for instance:

$$\mathbf{R} \text{ such as } \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{P}_\gamma = \gamma\mathbf{Id} + (1-\gamma)\mathbf{P} \tag{83}$$

Besides, the threshold function is taken linear $k(d) = k_0 + hd$. The constitutive equations then read:

$$\boldsymbol{\sigma} = (1 - d)\mathbf{E}^0 \cdot \boldsymbol{\varepsilon} \tag{84}$$

$$\dot{d} \geqslant 0, \quad Y - \gamma h d - (1-\gamma) h \bar{d} - k^0 \leqslant 0, \quad \dot{d}(Y - \gamma h d - (1-\gamma) h \bar{d} - k^0) = 0$$
$$\text{with} \begin{cases} Y = \frac{1}{2} \boldsymbol{\varepsilon} \cdot \mathbf{E}^0 \cdot \boldsymbol{\varepsilon} \\ \bar{d} = \mathbf{P} d \end{cases} \tag{85}$$

That corresponds to the model of Benvenuti et al. (2002), except for the fact that they chose a regularising operator which is not self-adjoint (while they considered a self-adjoint one would have been highly desirable).

To examine the localisation control, the tangent operator is calculated:

$$\dot{\boldsymbol{\sigma}} = \mathbf{E} \cdot (\dot{\boldsymbol{\varepsilon}} - \mathbf{P}_\gamma^{-1} \dot{\boldsymbol{\varepsilon}}) + \mathbf{H} \cdot \mathbf{P}_\gamma^{-1} \dot{\boldsymbol{\varepsilon}} \tag{86}$$

We recognise the expression which has often been obtained in this article, except for the fact that the regularising operator that appears in the rate equation is now the *inverse* of $\mathbf{P}_\gamma$. Thanks to Appendix A, Proposition 5, this inverse can be calculated (with L_c the characteristic length of $\mathbf{P}$ and $L_c\sqrt{\gamma}$ the one that appears in the inverse):

$$\mathbf{P}_\gamma^{-1} = \frac{1}{\gamma} \mathbf{Id} - \frac{1-\gamma}{\gamma} \mathbf{P}^{(L_c\sqrt{\gamma})} \tag{87}$$

Applying the results of Appendix B, effective regularisation is brought within the model as soon as:

$$0 \leqslant \frac{1}{\gamma} < 1 \Rightarrow \gamma > 1 \tag{88}$$

Even though the mathematical properties of $\mathbf{P}_\gamma$, and especially coerciveness, are preserved even for $\gamma \geqslant 1$ (we recall that a weighted average of a local and a non-local contribution leads to $0 < \gamma < 1$), the physics of (83) is questionable since the local part of the smoothed variable is strengthened while the non-local one is even subtracted: no interpretation in terms of a mixture of local and non-local variable is available anymore. This is in agreement with the results of Benvenuti et al. (2002) since they required a negative non-local hardening term. [4] Moreover, as the operator $\mathbf{P}_\gamma^{-1}$ is coercive, the control of the localisation band width is lost when damage becomes too brittle, see again Appendix B: in this model, that corresponds to high value of the damage variable, as observed again by Benvenuti et al. (2002).

Finally, it appears that a slight change in the non-local approach (smoothing one internal variable instead of the other, while they are equal in the local model) results in totally different conclusions. It seems that when the assumptions summed up in Section 2.3 are not fulfilled, no general results are available and the efficiency of the regularisation becomes highly dependent on the model.

7. Summary and conclusion

It has been shown that the format of generalised standard materials allows to express both the constitutive and the equilibrium equations as a variational principle: solutions are obtained as the minimum of a potential energy with respect to the strain field and the internal variable field. Taking advantage of this energetic formulation, non-local models of various types have been derived by replacing into the energy some of the local variables by smoothed counterparts, thus retrieving several proposals of the literature. The benefits of a single framework rely in the systematic exploration of the properties of these models, with

[4] Actually, these authors led their localisation analysis on a variant of this model where the damage function is quadratic in the stress–strain relation. Nevertheless, it seems that the conclusions are preserved in the case of a linear damage function.

regard to three aspects: their behaviour on strain-hardening constitutive laws, on strain-softening ones and their thermodynamic bases.

Under some assumptions, it has been shown that only two classes of non-local models effectively control strain localisation, but unfortunately, they lack from thermodynamic grounds. The first one relies on the replacement of the local internal variables by their smoothed counterparts within the stress–strain relation, while the second one consists in using a smoothed strain field when computing the evolution of the internal variables. Therefore, both of them may appear as algorithmic ad hoc schemes more than physics-based formulations. Nevertheless, the number of devoted publications for nearly twenty years now clearly demonstrate their potencies to deal with effective structural computations.

If thermodynamic grounds are looked for in addition to effective regularisation, the assumptions of our analysis have to be given up. But in that case, no general results seem to arise. Worse, minimal differences between models result in quite different behaviour. In that context, Comi and Perego's non-local model (2001) probably appears as the best choice for brittle damage, even though its numerical implementation may not be efficient according to Jirasek and Patzak (2002). However, its application to other types of laws is not systematic at all.

In conclusion, we think that a refinement of the potential energy by introducing strain gradients or internal variable gradients instead of replacing local quantities by smoothed ones appears as a quite more promising approach. Indeed, general frameworks can be obtained which guaranty satisfactory properties for strain-hardening as well as strain-softening laws, rely on thermodynamics and lead to efficient numerical treatments.

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Appendix A. Mathematical properties of the penalty regularising operator

For completeness, some mathematical results are presented here. They concern mainly the regularising operator $\mathbf{P}$ which is characterised by any of the three following propositions:

$$\begin{aligned}\mathbf{P} &: L^2(\Omega) \to L^2(\Omega)\\ &\quad u \mapsto \bar{u}\end{aligned}$$

Differential system

$$\bar{u} \text{ solution of } \begin{cases} \bar{u} - L_c^2 \Delta \bar{u} = u & \text{in } \Omega \\ \nabla \bar{u} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases} \tag{A.1}$$

Variational formulation

$$\begin{cases} \bar{u} \in H^1(\Omega) \\ \forall v \in H^1 \quad \int_\Omega \bar{u} v + L_c^2 \nabla \bar{u} \nabla v = \int_\Omega u v \end{cases} \tag{A.2}$$

Minimisation of a functional

$$\bar{u} = \underset{v \in H^1(\Omega)}{\arg\min} \left[\int_\Omega (v-u)^2 + L_c^2 (\nabla v)^2 \right] \tag{A.3}$$

The latter characterisation grounds the denomination "gradient penalty operator" since this regularising operator appears as a least square approximation of the original field u with a penalty term that limits high gradients.

This operator can be cast in the general context of convolution operators by choosing as weighting function ω Green's function G of the differential system (A.1):

$$(Pu)(x) = \int_{\Omega} G(x,y)u(y)\,\mathrm{d}y \tag{A.4}$$

In the case of 3D infinite medium, the Green function associated to the gradient penalty operator is easily deduced from the well-known Green function for Helmholtz' equation:

$$G(x,y) = \frac{1}{4\pi}\frac{\exp(-(\|x-y\|)/L_c)}{\|x-y\|} \tag{A.5}$$

For 1D situations, with body domain $\Omega =]0,1[$, Green's function $G(x,y)$ is plotted in Fig. 3 for several values of x and reads:

$$G(x,y) = \begin{cases} \dfrac{1}{L_c}\left(\dfrac{1}{\tanh\frac{1}{L_c}}\cosh\dfrac{y}{L_c} - \sinh\dfrac{y}{L_c}\right)\cosh\dfrac{x}{L_c} & \text{if } x \leqslant y \\ \dfrac{1}{L_c}\left(\dfrac{1}{\tanh\frac{1}{L_c}}\cosh\dfrac{x}{L_c} - \sinh\dfrac{x}{L_c}\right)\cosh\dfrac{y}{L_c} & \text{if } x \geqslant y \end{cases} \tag{A.6}$$

To simplify the following demonstrations, we assume from now on that $L_c = 1$, without lack of generality. The following notations are introduced: (u,v) denotes the usual dot product in $L^2(\Omega)$ while $\|u\| = (u,u)^{1/2}$ is the associated norm.

Proposition 1. *The gradient penalty operator* **P** *fulfils the following properties*:

(a) **P** *is an application whose range is included in* $H^1(\Omega)$
(b) **P** *is a positive operator*
(c) **P** *is linear and injective*
(d) **P** *is self-adjoint*

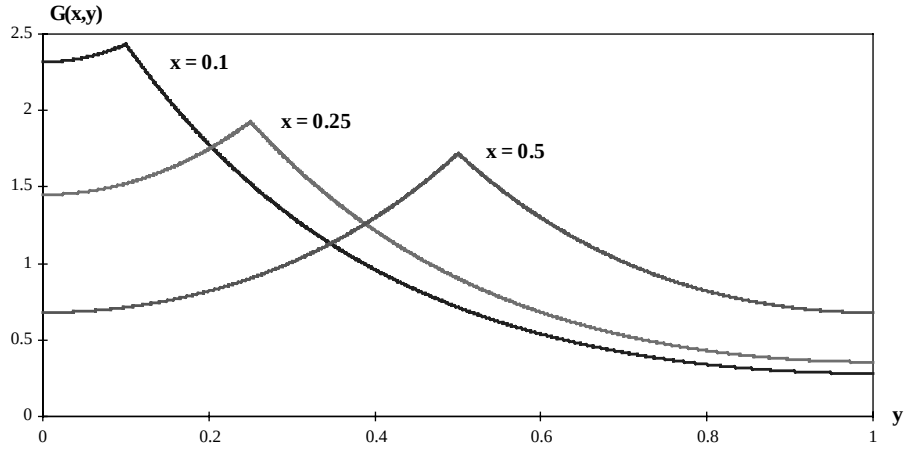

Fig. 3. Green function associated to operator **P** ($L_c^2 = 0,1$) on the line segment $]0,1[$.

(e) **P** *leaves unchanged the mean value*:

$$\forall u \in L^2(\Omega) \quad \int_\Omega \mathbf{P}u = \int_\Omega u$$

(f) *The constant functions are left unchanged by operator* **P**. *Respectively, if* $\mathbf{P}u$ *is constant, then* $u = \mathbf{P}u$ *(constant)*

(g) **P** *is continuous in* $L^2(\Omega)$

(h) *The principle of maximum holds true for operator* **P**:

$$\text{inf ess } u \leqslant \mathbf{P}u \leqslant \text{sup ess } u \quad \begin{cases} \text{inf ess } u = \sup\{ C \in IR; \quad C \leqslant u \quad \text{a.e.} \} \\ \text{sup ess } u = \inf\{ C \in IR; \quad C \geqslant u \quad \text{a.e.} \} \end{cases}$$

Proof

Property (a). We refer to the characterisation (A.3) for operator **P**. Thanks to the properties of the function to minimise (strict convexity, continuity and coerciveness in $H^1(\Omega)$ as soon as $u \in L^2(\Omega)$, the problem (A.3) admits a unique solution $\bar{u}$ which belongs to $H^1(\Omega)$.

Property (b). We refer to the variational formulation (A.2) and take as test function $v = \mathbf{P}u$:

$$(u, \mathbf{P}u) = (\mathbf{P}u, \mathbf{P}u) + (\nabla \mathbf{P}u, \nabla \mathbf{P}u) \geqslant 0$$

Property (c). The linearity results directly from the characterisation (A.2). To prove the injectivity, we show that ker P $= \{0\}$. The characterisation (A.2) again leads to:

$$P(u) = 0 \Rightarrow \forall v \in H^1(\Omega) \quad \int_\Omega uv = 0$$

As $H^1(\Omega)$ is dense in $L^2(\Omega)$, the following equation also holds:

$$\forall v \in L^2(\Omega) \quad \int_\Omega uv = 0$$

Besides, u belongs to $L^2(\Omega)$, so that the previous equation proves $u = 0$.

Property (d). **P** is self-adjoint if and only if:

$$\forall u, v \in L^2(\Omega) \quad (u, \mathbf{P}v) = (\mathbf{P}u, v)$$

This result can be obtained by casting $\mathbf{P}v$ and $\mathbf{P}u$ as test functions in the characterisation (A.2) of $\mathbf{P}u$ and $\mathbf{P}v$ respectively:

$$\begin{cases} \int_\Omega \mathbf{P}u\mathbf{P}v + \nabla \mathbf{P}u \nabla \mathbf{P}v = \int_\Omega u\mathbf{P}v \\ \int_\Omega \mathbf{P}v\mathbf{P}u + \nabla \mathbf{P}v \nabla \mathbf{P}u = \int_\Omega v\mathbf{P}u \end{cases} \Rightarrow \int_\Omega u\mathbf{P}v = \int_\Omega v\mathbf{P}u$$

Property (e). The result is readily obtained by applying the variational formulation (A.2) with test function $v = 1$.

Property (f). First, let us denote J the function to minimise in (A.3):

$$J(v) = \int_\Omega (v - u)^2 + (\nabla v)^2$$

Consider that $u = c$ is a constant function. Then, $J(c) = 0$. Moreover, it can be noticed that:

$$\forall v \in H^1(\Omega) \quad J(v) \geqslant 0$$

It proves that c is the (unique) minimum of J, that is $\mathbf{P}c = c$.

Respectively, consider that $\mathbf{P}u = c$ is constant. As we know now that $\mathbf{P}c = c$, the injectivity of $\mathbf{P}$ allows to conclude that $u = c$.

Property (g). First, by taking $v = \mathbf{P}u$ in the variational formulation (A.2), one gets

$$\|\mathbf{P}u\|^2 + \|\nabla \mathbf{P}u\|^2 = (\mathbf{P}u, u)$$

A straightforward application of Cauchy–Schwartz inequality leads to the conclusion

$$\|\mathbf{P}u\|^2 \leqslant \|\mathbf{P}u\|^2 + \|\nabla \mathbf{P}u\|^2 = (\mathbf{P}u, u) \leqslant \|\mathbf{P}u\|\|u\| \Rightarrow \|\mathbf{P}u\| \leqslant \|u\|$$

It can be noticed that this upper bound is optimal since the equality is reached for constant fields.

Property (h). See Brezis (1983). □

Proposition 2. *The function $u \mapsto \|\mathbf{P}u\|$ is not coercive in $L^2(\Omega)$.*

Proof. To prove this negative result, a counter example will be exhibited with $\Omega =]0, 1[$. More precisely, we build a sequence (φ_n) such as, on one hand, $\mathbf{P}\varphi_n \to 0$ and, on the other hand, there exists $\alpha > 0$ such as $\|\varphi_n\| \geqslant \alpha$. Such a sequence proves the lack of coerciveness, since:

$$\text{with } u_n = \frac{\varphi_n}{\|\mathbf{P}\varphi_n\|}, \quad \|u_n\| = \frac{\|\varphi_n\|}{\|\mathbf{P}\varphi_n\|} \to \infty \quad \text{while } \|\mathbf{P}u_n\| = \frac{\|\mathbf{P}\varphi_n\|}{\|\mathbf{P}\varphi_n\|} = 1 < \infty$$

Now, let us show that such a sequence (φ_n) can be given by:

$$\varphi_n(x) = \sqrt{n}\mathrm{e}^{-nx} \quad \text{with } n \geqslant 2 \tag{A.7}$$

First, consider the norm of φ_n:

$$\|\varphi_n\| = \sqrt{\int_0^1 n\mathrm{e}^{-2nx}} = \sqrt{\frac{1}{2}(1 - \mathrm{e}^{-2n})} \geqslant \alpha = \sqrt{\frac{1}{2}(1 - \mathrm{e}^{-4})} > 0$$

Then, the regularised sequence $(\mathbf{P}\varphi_n)$ has to be derived. Solving the differential equation (A.1) leads to:

$$(\mathbf{P}\varphi_n)(x) = \frac{\sqrt{n}}{1 - n^2}\left(\mathrm{e}^{-nx} - \frac{n}{\sinh 1}\cosh(1 - x) + \frac{n\mathrm{e}^{-n}}{\sinh 1}\cosh x\right)$$

Finally, $\mathbf{P}\varphi_n$ goes indeed to zero:

$$\|\mathbf{P}\varphi_n\| = An^{-1/2} + o(n^{-1/2}) \underset{n\to\infty}{\to} 0 \qquad \square$$

Proposition 3. *The function $u \mapsto \|\mathbf{M}u\|$ is not coercive in $L^2(\Omega)$ for bounded functions ω. This is also true for integrable singular weighting function of the form $\omega(x, y) = \|x - y\|^{-\alpha}$, but the proof is more tedious and will not be detailed in this paper.*

Proof. We consider $\Omega =]0, 1[$ and the sequence (φ_n) as defined in (A.7):

$$\varphi_n(x) = \sqrt{n}\mathrm{e}^{-nx} \quad \text{with } n \geqslant 2$$

Then, $\mathbf{M}\varphi_n$ is equal to:

$$(\mathbf{M}\varphi_n)(x) = \sqrt{n}\int_0^1 \exp(-ny)\omega(x,y)\,\mathrm{d}y$$

As ω is bounded, say $|\omega(x,y)| \leqslant B$, then $\mathbf{M}\varphi_n$ is also bounded by:

$$|(\mathbf{M}\varphi_n)(x)| \leqslant B\sqrt{n}\int_0^1 \mathrm{e}^{-ny}dy = Bn^{-1/2}(1-\mathrm{e}^{-n})$$

Finally, it can be shown that $\mathbf{M}\varphi_n$ goes to zero, and thus $\mathbf{M}$ is not coercive:

$$\|\mathbf{M}\varphi_n\| \leqslant Bn^{-1/2}(1-\mathrm{e}^{-n}) \underset{n\to\infty}{\to} 0 \qquad \square$$

Proposition 4. *For any positive operator* $\mathbf{R}$ *of* $L^2(\Omega)$, $\mathbf{R}_\gamma = \gamma\mathbf{Id} + (1-\gamma)\mathbf{R}$ *is coercive as soon as* $0 < \gamma < 1$.

Proof. For $0 < \gamma < 1$, we have the following inequality since $\mathbf{R}$ is positive:

$$\forall u \in L^2(\Omega) \quad \|\mathbf{R}_\gamma u\|^2 = \gamma^2\|u\|^2 + (1-\gamma)^2\|\mathbf{R}u\|^2 + 2\gamma(1-\gamma)(u,\mathbf{R}u) \geqslant \gamma^2\|u\|^2$$

It is then straightforward that:

$$\forall u \in L^2(\Omega) \quad \|u\| \to \infty \Rightarrow \|\mathbf{R}_\gamma u\| \to \infty \qquad \square$$

Proposition 5. *Consider the operator* $\mathbf{P}$ *with internal length* L_c, *denoted by* $\mathbf{P}^{(L_c)}$. *Then, the operator* $\mathbf{P}_\gamma = \gamma\mathbf{Id} + (1-\gamma)\mathbf{P}^{(L_c)}$ *is invertible when* $0 < \gamma < 1$ *with inverse:*

$$\mathbf{P}_\gamma^{-1} = \frac{1}{\gamma}\mathbf{Id} - \frac{1-\gamma}{\gamma}\mathbf{P}^{(L_c\sqrt{\gamma})}$$

Proof. Let us denote $w = \mathbf{P}_\gamma u$ and $\bar{u} = \mathbf{P}^{(L_c)}u$. Then, the characterisation (A.1) leads to:

$$\begin{cases} w = \gamma u + (1-\gamma)\bar{u} = \gamma(\bar{u} - L_c^2\Delta\bar{u}) + (1-\gamma)\bar{u} = \bar{u} - \gamma L_c^2\Delta\bar{u} \\ \dfrac{\partial\bar{u}}{\partial n} = 0 \end{cases} \Rightarrow \bar{u} = \mathbf{P}^{(L_c\sqrt{\gamma})}w$$

The conclusion is immediate:

$$u = \mathbf{P}_\gamma^{-1}w = \frac{w-(1-\gamma)\bar{u}}{\gamma} = \left[\frac{1}{\gamma}\mathbf{Id} - \frac{1-\gamma}{\gamma}\mathbf{P}^{(L_c\sqrt{\gamma})}\right]w \qquad \square$$

Appendix B. Localisation analysis

In this appendix, we present an analysis to measure the effective regularisation brought by non-local models. It is based on an estimation of the localisation band width selected by non-local models and it is achieved by means of a bifurcation analysis in the same way as Pijaudier-Cabot and Benallal (1993) did for a brittle elastic non-local damage model. Therefore, we restrict our attention to an infinite homogeneous incrementally linear and rate-independent medium. It results in the following assumptions:

- bifurcation from a homogeneous state $(\boldsymbol{\varepsilon}^0, a^0)$,
- the boundary conditions are not taken into account,
- the real constitutive relation is replaced by the linear comparison solid.

Under these assumptions, it has been showed that most of the regularisation methods examined in the article lead to rate constitutive laws of the form:

$$\dot{\boldsymbol{\sigma}} = \mathbf{H} \cdot \mathbf{R}\dot{\boldsymbol{\varepsilon}} + \mathbf{E} \cdot (\dot{\boldsymbol{\varepsilon}} - \mathbf{R}\dot{\boldsymbol{\varepsilon}}) \tag{B.1}$$

where $\mathbf{H}$ and $\mathbf{E}$ denote respectively the tangent operator for the local model and the elastic operator for the constitutive rate law. Their expressions for the rate-independent model (1)–(3) read:

$$\mathbf{E} = \frac{\partial^2 \Phi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}}, \qquad \mathbf{H} = \mathbf{E} + \frac{1}{\dfrac{\partial f}{\partial A}\dfrac{\partial^2 \Phi}{\partial a \partial a}\dfrac{\partial f}{\partial A}} \left(\frac{\partial^2 \Phi}{\partial \boldsymbol{\varepsilon} \partial a} \frac{\partial f}{\partial A} \right) \otimes \left(\frac{\partial f}{\partial A} \frac{\partial^2 \Phi}{\partial a \partial \boldsymbol{\varepsilon}} \right) \tag{B.2}$$

We claim that thanks to the definite positive character of the elastic tensor $\mathbf{E}$, the second term in the right hand side of (B.1) brings some additional stability compared to the purely local model. To prove this assertion, we carry on the localisation analysis in the special case of a regularising operator $\mathbf{R}$ based on the gradient penalty operator $\mathbf{P}$:

$$\mathbf{R} = \mathbf{P}_\gamma = \gamma \mathbf{Id} + (1 - \gamma)\mathbf{P} \tag{B.3}$$

Consider a velocity perturbation in the direction $\mathbf{n}$ of the form:

$$\mathbf{v}(\mathbf{x}) = \mathbf{A}\varphi_{\xi\mathbf{n}}(\mathbf{x}) \quad \text{with } \varphi_{\xi\mathbf{n}}(\mathbf{x}) = \exp(-\mathrm{i}\xi \mathbf{n} \cdot \mathbf{x}) \tag{B.4}$$

where ξ is the wave number, $\mathbf{A}$ the amplitude of the perturbation and i the imaginary constant ($\mathrm{i}^2 = -1$). The corresponding strain rate is:

$$\dot{\boldsymbol{\varepsilon}}(\mathbf{x}) = -\frac{\mathrm{i}\xi}{2}(\mathbf{A} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{A})\varphi_{\xi\mathbf{n}}(\mathbf{x}) \tag{B.5}$$

As the boundary conditions are neglected, such perturbations are eigenvectors of $\mathbf{P}_\gamma$:

$$\mathbf{P}_\gamma \varphi_{\xi\mathbf{n}}(\mathbf{x}) = \lambda_\xi \varphi_{\xi\mathbf{n}}(\mathbf{x}) \quad \text{with } \lambda_\xi = \gamma + \frac{1 - \gamma}{1 + L_c^2 \xi^2}, \quad \gamma < \lambda_\xi \leqslant 1 \tag{B.6}$$

with L_c the characteristic length of $\mathbf{P}$. Substituting (B.5) and (B.6) in (B.1) and writing the inner equilibrium equations for the stress rate leads to:

$$\mathrm{div}\dot{\boldsymbol{\sigma}} = \xi^2 \lambda_\xi [\mathbf{n}\mathbf{H}^*(\xi)\mathbf{n}]\mathbf{A} = 0 \quad \text{with} \begin{cases} \mathbf{H}^*(\xi) = (\mathbf{H} + \omega_\gamma(\xi)\mathbf{E}) \\ \omega_\gamma(\xi) = \dfrac{1 - \lambda_\xi}{\lambda_\xi} = (1 - \gamma)\dfrac{L_c^2 \xi^2}{1 + \gamma L_c^2 \xi^2}, \quad 0 \leqslant \omega_\gamma(\xi) < \dfrac{1 - \gamma}{\gamma} \end{cases} \tag{B.7}$$

where $\omega_\gamma(\xi)$ measures the stabilising influence of the elasticity tensor $\mathbf{E}$ for a perturbation of wave number ξ, as plotted Fig. 4. It can be noticed that this stabilising influence is higher for smaller coerciveness parameter γ, even becoming infinite for non-coercive operators ($\gamma = 0$). Finally, the rate problem admits non-trivial solutions if and only if:

$$\exists \xi, \mathbf{n} \quad \det\,[\mathbf{n}\mathbf{H}^*(\xi)\mathbf{n}] = 0 \Longleftrightarrow \exists \xi, \mathbf{n} \quad \mu^*(\xi, \mathbf{n}) \leqslant 0 \tag{B.8}$$

with $\mu^*(\xi, \mathbf{n})$ the smallest eigenvalue of $\mathbf{n}\mathbf{H}^*(\xi)\mathbf{n}$.

To draw some conclusions, we will also denote $\mu_1(\mathbf{n}) \leqslant \mu_2(\mathbf{n}) \leqslant \mu_3(\mathbf{n})$ the eigenvalues of $\mathbf{n}\mathbf{H}\mathbf{n}$ and $e_1(\mathbf{n}) \leqslant e_2(\mathbf{n}) \leqslant e_3(\mathbf{n})$ the eigenvalues of $\mathbf{n}\mathbf{E}\mathbf{n}$, so that the condition for localisation in the *local* medium reads:

$$\exists \mathbf{n} \quad \mu_1(\mathbf{n}) \leqslant 0 \tag{B.9}$$

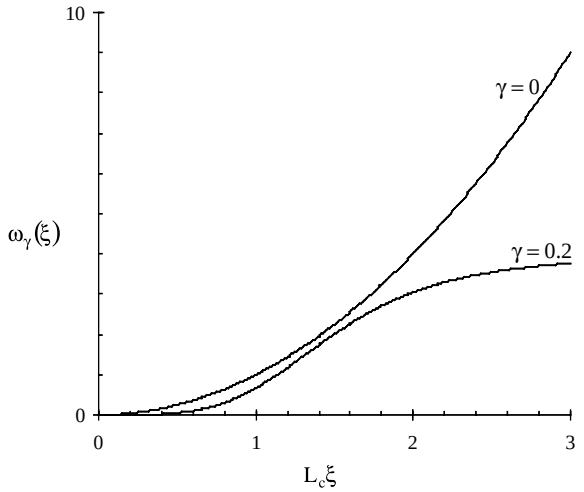

Fig. 4. Stabilising influence of the elasticity tensor for a given perturbation.

We have the following properties (see for instance Wilkinson, 1965):

$$\mu^*(\xi = 0, \mathbf{n}) = \mu_1(\mathbf{n}) \tag{B.10}$$

$$\mu^*(\xi, \mathbf{n}) \geqslant \mu_1(\mathbf{n}) + \underbrace{\omega_\gamma(\xi)}_{\geqslant 0}\underbrace{e_1(\mathbf{n})}_{>0} \geqslant \mu_1(\mathbf{n}) \tag{B.11}$$

$$\mu^*(\xi, \mathbf{n}) \leqslant \mu_1(\mathbf{n}) + \omega_\gamma(\xi) e_3(\mathbf{n}) \tag{B.12}$$

Finally, the conclusions of Pijaudier-Cabot and Benallal (1993) are retrieved:

- The criterion for the occurrence of bifurcations obtained for the non-local model is similar to that for the corresponding local one. Moreover, the condition for localisation in the local continuum is a lower bound of the criterion for bifurcation in the non-local continuum, thanks to (B.10) and (B.11).
- For $\gamma = 0$ (non-coercive regularising operator), the wave lengths $2\pi/\xi$ for localised perturbations cannot be below a certain critical value, thanks to (B.11) and the fact that ω_0 goes to infinity with smaller wave lengths:

$$\begin{aligned} &\forall \xi, \mathbf{n} \quad \mu^*(\xi, \mathbf{n}) \geqslant \mu_1(\mathbf{n}) + \omega_0(\xi) e_1(\mathbf{n}) \geqslant \min_{\mathbf{n}} \mu_1(\mathbf{n}) + \omega_0(\xi) \underbrace{\min_{\mathbf{n}} e_1(\mathbf{n})}_{>0} \\ &\qquad\qquad \Downarrow \\ &\forall \xi > \xi_c, \mathbf{n} \quad \mu^*(\xi, \mathbf{n}) > 0 \quad \text{with } \omega_0(\xi_c) = -\frac{\min_{\mathbf{n}} \mu_1(\mathbf{n})}{\min_{\mathbf{n}} e_1(\mathbf{n})} \end{aligned} \tag{B.13}$$

- For $\gamma \neq 0$ and sufficiently brittle materials, the wave lengths $2\pi/\xi$ for localised perturbations may go to zero because of (B.12), which hints at preferring non-coercive regularising operator ($\gamma = 0$) to perform regularisation:

$$\begin{aligned} &\forall \xi, \mathbf{n} \quad \mu^*(\xi, \mathbf{n}) \leqslant \mu_1(\mathbf{n}) + \omega_\gamma(\xi) e_3(\mathbf{n}) \leqslant \mu_1(\mathbf{n}) + \frac{1-\gamma}{\gamma} e_3(\mathbf{n}) \\ &\qquad\qquad \Downarrow \\ &\text{if } \exists \mathbf{n_c} \quad \mu_1(\mathbf{n_c}) \leqslant -\frac{1-\gamma}{\gamma} e_3(\mathbf{n_c}) \quad \text{then } \forall \xi \ \ \mu^*(\xi, \mathbf{n_c}) \leqslant 0 \end{aligned} \tag{B.14}$$

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